Title:
Notes on the CAD-Compatible Conversion of Multi-Sided Surfaces

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Introduction:

The majority of aesthetic objects is represented by free-form shapes, and modeling these naturally involves multi-sided (i.e., non-four-sided) surfaces, as well. The mathematical representation of such patches is still an active area in CAGD, and although a great variety of approaches have been published, none of the genuine n-sided formulations have been standardized so far.

On the other hand, commercial CAD/CAM systems and related application programs only accept data in standard formats, such as tensor product NURBS surfaces. For this reason, it is a widely applied practice to convert multi-sided surfaces into a CAD-compatible representation either by (i) approximating them with larger quadrilaterals, *trimming* away the exterior part beyond the boundaries, or (ii) *splitting* them into smaller four-sided patches.

Both techniques have their deficiencies. They only approximate the original multi-sided surface, and trimming – in general – cannot ensure even C^0 continuity between the adjacent patches. In the splitting scheme the subdividing curves in the interior weaken the overall continuity of the surface.

Ideally, we would like to have an n-sided patch that:

i) can be used for design (has intuitive controls),

ii) can be attached to adjacent patches with G^1 or higher continuity,

iii) and can also be represented *accurately* as a tensor product NURBS surface.

The above problem can be resolved, if the multi-sided surfaces can be represented as rational polynomials of two parametric variables. Then they can be directly converted into NURBS form, without either changing the surfaces or harming continuity. The result will be a collection of *watertight trimmed surfaces*.

Some of the well-known multi-sided schemes allow computing a trimmed bi-parametric representation. Our goal in this paper is to review these and discuss the difficulties of the conversion process. We are going to provide further insights into specific computational and geometric problems, that have not been discussed elsewhere and are useful for analyzing the "pros and cons" of these representations. We will investigate four schemes, and show actual high-degree conversion examples with control grids. A general discussion will conclude the paper.

S-patch:

The S-patch of Loop & DeRose [4] is a generalization of the Bézier triangle, or, more precisely, a Bézier simplex mapping from $(n-1)$D to 3D, where the n coordinates of the domain are supplied by generalized barycentric coordinates. The *depth* (d) of an S-patch is the number of deCasteljau steps it takes to evaluate a surface point, i.e., something similar to the degree of a Bézier triangle (but not the degree of the S-patch itself). S-patches have many nice properties, and are known to be convertable into tensor product rational Bézier surfaces of degree $d(n-2)$.

One drawback of this representation is its large number of control points, which renders it inconvenient for interactive design. For example, a five-sided patch of depth 5 has 126 control points, while a 4-sided tensor product patch has only 36. A possible workaround is to use a G^1 *frame* for design that defines the tangent planes at the boundaries. After increasing the depth by 3, these boundary constraints can be interpolated [5], and the remaining interior control points can be set by some heuristic to generate a smooth surface [6], see the figure below.

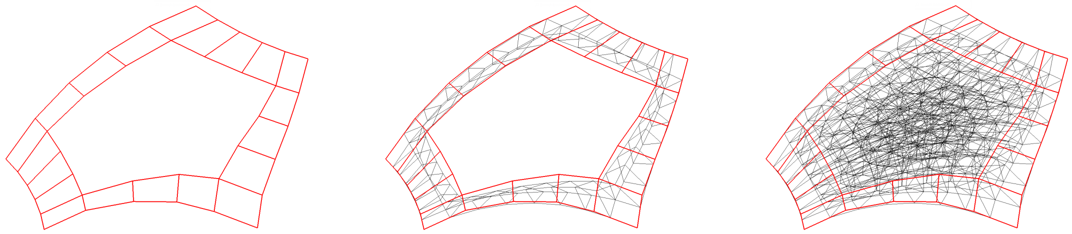


Fig. 1: Creating an S-patch from a G^1 frame.

The CAD-compatible conversion presented in [4] is a two-step process: first convert the surface into a four-sided S-patch, and then to a tensor product patch. The first step is based on the composition of Bézier simplexes, which has very high complexity. Even using an efficient algorithm [2], converting a modest-sized S-patch still requires minutes of computation on today's machines [7].

Here we propose an alternative conversion process. Since a Bézier simplex is just a polynomial, the only problem is how to express the generalized barycentric coordinates as a rational polynomial of the (u,v) parameters on the 2D domain. Using Wachspress coordinates over a regular n-sided polygon, the barycentric coordinates $\{\lambda_i\}$ are expressed as

$$\lambda_i(u,v) = \prod_{\substack{j=1 \\ j\notin\{i-1,i\}}}^{n} h_j(u,v) \Bigg/ \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\notin\{k-1,k\}}}^{n} h_j(u,v), \tag{2.1}$$

where the indexing is cyclic, and $h_j(u,v)$ is a distance function from the j-th side of the domain polygon. The implicit equation of the line containing a given side is suitable for this purpose, and is also a linear polynomial, so the Wachspress coordinates can be expressed as rational polynomials of degree $n-2$. We normalize the distances such that they take on the value 1 at vertices adjacent to the side.

With this method, the Bézier control points of the tensor product representation can be located by straightforward computation, which takes only milliseconds.

Warren's patch:

Warren [8] created multi-sided patches from Bézier triangles by assigning 0/0 *base points* to some of the control points, essentially cutting off the corners, and thus creating 5- and 6-sided surfaces. A simple conversion to a (degenerate) tensor product form is also shown in the paper.

A nice property of this patch is that the "remaining" control points define the behavior of the boundary in the same way as in a normal Bézier triangle, i.e., the first control row defines its position as a Bézier curve, the second its first derivatives etc.

Fig. 2: Warren's 5-sided patch and its conversion to NURBS.

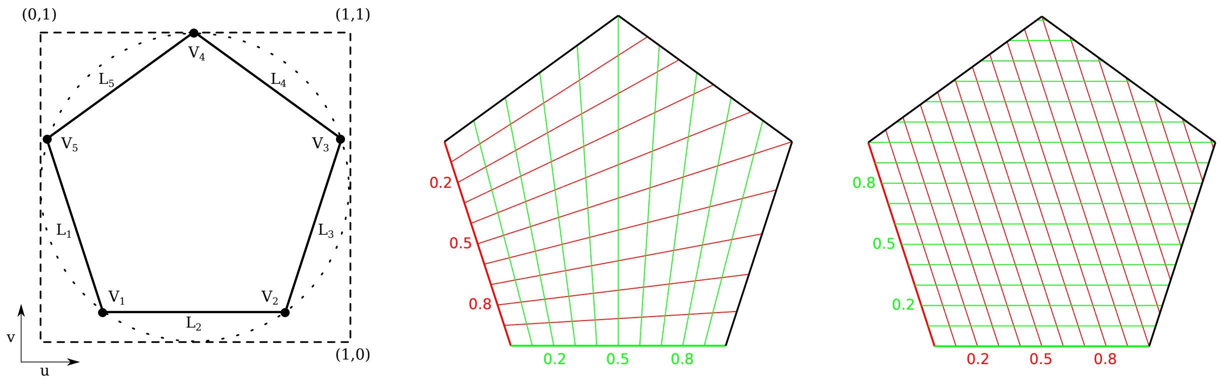
Note, however, that not all degree configurations are available. A 6-sided patch with degree-d boundaries can be created from a triangle of degree $3d$, but due to its asymmetric construction, a 5-sided patch cannot have boundaries of the same degree. Moreover, using control points with zero weight is not a standard practice, and is not supported by many systems. Meshing also presents a problem, as a uniform grid on the domain would result in distorted triangles (the "trimmed" sides correspond to corners).

Kato's patch:

Kato [3] proposed a surface defined as the transfinite interpolation of boundary curves with cross-derivatives. When these boundary constraints are given as a G^1 frame (and hence are polynomial), the whole patch may become polynomial. The tricky part is the parameterization: this representation uses two local parameters, a *side parameter* s_i that takes on values from 0 to 1 as it sweeps from one adjacent side to the other, and a *distance parameter* h_i that vanishes on the base side i, see Figure 3.

The distance parameters can be computed the same way as in the creation of S-patches, and for side parameters we can use $s_i = h_{i-1}/(h_{i-1} + h_{i+1})$, which gives a rational polynomial representation.

Kato's patch is easily extendable to handle G^2 continuity, if the side constraints have 3 control rows (so the cross-degree $d^\perp$ is 2 instead of 1). Then the whole surface becomes a rational tensor product Bézier patch of degree $nd + (n-1)(d^\perp + 1) + d^\perp$.



(a) Domain (b) s_i isolines for two sides (c) h_i isolines for two sides

Fig. 3: Parameterization.

Charrot–Gregory patch:

The same idea can be used to convert Charrot–Gregory patches [1] that use only side parameters. (Note that on a regular domain the s_i parameters defined above will be the same as the radial parameterization in the original paper.) The input is given as a G^1 frame; the converted patch is of degree $nd + 2(n-2)$.

For triangular surfaces, we can use the h_{i-1} parameter as a side parameter for the i-th side instead of s_i, thereby reducing the overall degree to $d + 3$.

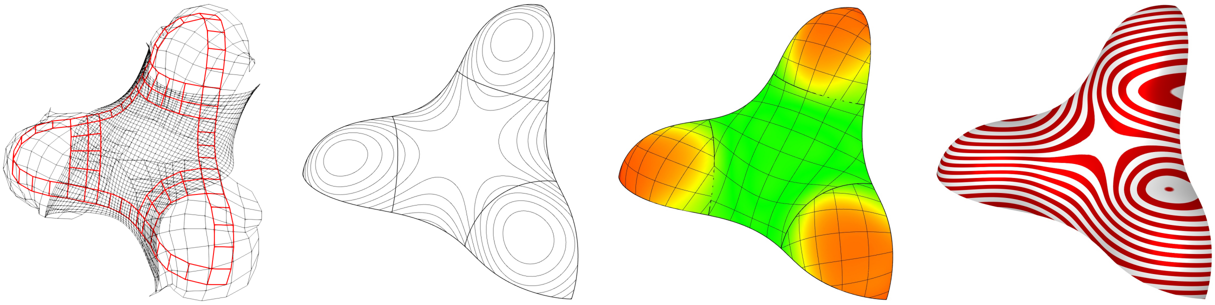


Fig. 4: A model with four trimmed patches shown with contours, mean curvature and isophote lines.

Discussion:

One aspect of the tensor product conversion we have not touched on before is the *quality* of the control net. Aside from Warren's patch, which has singular control points, all the other representations have singularities in or outside their domains. When a singular point is close to the domain of the tensor product patch (i.e., the unit square), the control points in the vicinity show erratic behavior.

Kato's patch is singular at the corner vertices, and the S-patch is singular on the circle that goes through the intersections of the lines containing the domain edges, while the Charrot–Gregory patch is singular on a larger n-sided polygon touching the above circle from the outside. Practically this means that excluding the triangular S-patch (which is not rational) and the Charrot–Gregory patch for $n \leq 6$ (where singularities are relatively far away), all of these converted tensor product surfaces are likely to have badly oscillating control points (possibly tending to infinity), which may lead to numerical issues.

We present a solution to this problem. Normally the multi-sided domain is inside the unit square (Figure 3), so that the trimming curves will be inside the surface, but if we lift this constraint, we can create a larger multi-sided domain, thereby separating the unit square from the singularities. This means that the actual "trimmed" region will be outside the standard $[0,1]^2$ domain; this may not be supported by some applications, but the control structure will be close to the surface.

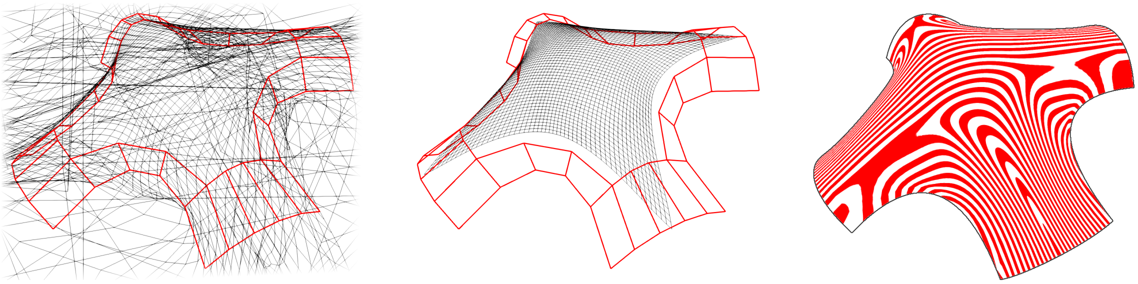


Fig. 5: An 8-sided Charrot–Gregory patch with default and enlarged domain, and isophote lines.

n	S-patch [4]	Warren [8]	Kato [3]	Charrot–Gregory [1]
3	$d[+3]$ (both Bézier triangles)		$3d+5$	$d+3$
5	$3d[+9]$	$\approx 3d$	$5d+9$	$5d+6$
6	$4d[+12]$	$3d$	$6d+11$	$6d+8$
7+	$(n-2)(d[+3])$	N/A	$nd+2n-1$	$nd+2n-4$

Tab. 1: Rational polynomial degrees of the converted surfaces for different number of sides, assuming boundary curves of degree d. Gray cells indicate that the surface is susceptible to the singularity issue. For S-patches, the number in brackets is applied when the surface is generated by a degree-d G^1 frame.

Table 1 summarizes the degrees of all four representations. It can be seen that these patches have relatively high degrees, in particular when the number of sides and the degree of the boundaries are raised. While Warren's patch outperforms the others in this respect, the use of base points somewhat limits its usability in CAD systems. Kato's surface always has singularities, and its degree is fairly high, but it is the only construction where G^2 continuity can be easily achieved. We found that while the Charrot–Gregory patch has a slightly higher degree than the S-patch, it has much lower computational cost in its multi-sided form, and have no control net quality problems for 5- and 6-sided configurations.

Conclusion:

In the full paper we are going to give representation details, and further analyze the difficulties of the conversion process. The computational efficiency of the proposed procedures and the avoidance of wiggling control structures will be the focal part of our discussion, together with several comparative examples.

Acknowledgment:

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