Transfinite Surface Patches Using Curved Ribbons

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Abstract

An important problem in Computer Aided Design is to create digital representations for complex free-form objects that produce nice, predictable shapes and facilitate real-time editing in 3D. The clue to curve network-based design is the construction of smoothly connected multi-sided patches. A new type of transfinite surface, called Composite Ribbon (CR) patch is introduced, that is a combination of curved ribbons and ensures G^1 continuity over non-regular, convex polygonal domains. After discussing the construction and the preferred parameterization scheme, a few simple examples conclude the paper.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Curve, surface, solid, and object representations

1. Introduction

Creating general topology free-form objects, composed of smoothly connected surface patches, is a fundamental problem in CAGD. Aesthetic appearance is crucial for a wide variety of models including cars, household appliances, office furniture, containers and many others. While the majority of such patches are four-sided, almost all industrial objects contain *general n-sided patches* that need to be inserted into some arrangement of quadrilaterals (e.g. Figure 1).

General topology surfacing is a tough problem, and all known techniques expose deficiencies. A very simple example is shown in Figure 2a, where a 3D network of adjacent 5- and 6-sided patches is to be edited and interpolated.

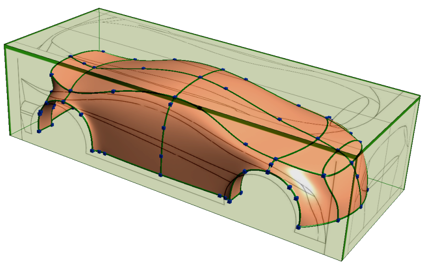
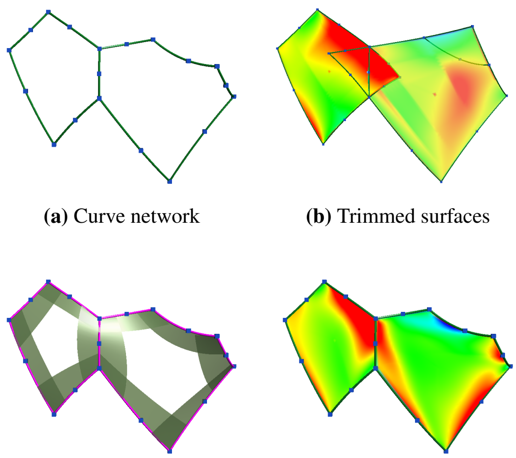


Figure 1: Concept car defined by orthogonal sketches.

The standard approach is to combine *trimmed* and *stitched* bi-parametric surfaces that yield a model with numerical continuity. The boundaries of the patches and the trimming curves have different representational form and design flexibility. Creating a truly symmetric three-sided patch is not possible in the four-sided domain. It is not obvious in Figure 2b how to extend the given curve segments and how to construct and edit the common middle trim curve with smooth connection. Another approach is splitting *n*-sided regions into smaller *quadrilateral tiles*, but adding an appropriate center point and internal subdividing curves may lead to unexpected curvatures.

Recursive subdivision surfaces, controlled by a general topology polyhedra, are used in many applications. These yield a set of smoothly connected quadrilaterals combined with *n*-sided surface patches, however, difficulties include the “ab initio” creation of good control polyhedra and the direct interpolation of curves with tangential constraints.

In this paper we explore a fourth approach, where the network automatically spans a collection of *multi-sided transfinite patches*. Feature curves come from 2D sketches or are defined in 3D. Editing the boundaries directly modifies the adjacent patches, connected in a watertight manner, thus users can focus on shape concepts and aesthetic requirements. In this approach the interior of the shape is solely defined by boundary ribbons, and there is no need to deal with a grid of interior control points. Transfinite patches also have their deficiencies — ribbons may not always meet



(a) Curve network **(b)** Trimmed surfaces

(c) Ribbon surfaces **(d)** Transfinite surfaces

Figure 2: Comparison of different modeling approaches.

user expectations, and standard surfaces can only be reproduced in approximate sense. Figure 2d illustrates the simplicity of curve network-based design. The network automatically defines the ribbons (Fig. 2c), and the transfinite patches (Fig. 2d). The curves can easily be modified and the topology redesigned, then the model will adjust accordingly.

Transfinite surface interpolation is a classical area of CAGD. Its origin goes back to the late 60's, when Coons formulated his Boolean sum surface [Coo67]. In the next two decades, several papers were published, first on triangular patches [Far02], and later on genuine n-sided patches, see Charrot and Gregory [CG84, Gre86], Sabin [Sab96], and Kato [Kat91]. The alternatives of creating n-sided transfinite patches have been recently reviewed in [VRS11].

In this paper we present the *Composite Ribbon* or CR patches. Unlike previous constructions, here we propose combining curved (i.e., non-linear) ribbons, and introduce a new ribbon parameterization using Wachpress coordinates. CR patches will be demonstrated through a few examples, and suggestions for future work conclude the paper. Details of the construction and related proofs can be found in [Sal12].

2. Composite ribbon patches

The CR patch is a transfinite surface interpolating $n \geq 3$ boundaries $P_i(s_i)$, $1 \leq i \leq n$, and related cross-derivative functions $T_i(s_i)$. The surface is defined as a combination of special *curved ribbons*, comprising the above functions. Let Γ be a convex polygon in the (u,v) domain plane, and map the sides of the polygon, Γ_i, onto the boundaries of the patch. The local side and distance parameters of the ribbons are computed from (u,v), i.e., $s_i = s_i(u,v)$, $d_i = d_i(u,v)$, and we associate a blending function $B_i(u,v) = B_i(d_1,\dots,d_n)$ to each side. To create a CR patch, the following constituents must be provided: (i) an n-sided domain polygon, (ii) blending functions, (iii) n ribbon surfaces and (iv) appropriate methods to parameterize the ribbons. For different domain creation methods, see [VRS11]. All the other aspects will be treated in the following sections, one by one.

2.1. Generalized blending functions

We need blending functions over the polygonal domain that reproduce the ribbons along their boundaries. These need to satisfy special interpolating properties. For each (u,v) point we determine an n-tuplet of distance values. Each d_i is associated with the i-th side: d_i is equal to 0 on side Γ_i, and it increases monotonically as we move away from Γ_i. In our patch formulations distance-based rational blending functions are used to combine ribbons. The basic requirement is that the blending function B_i is equal to 1 on Γ_i, and vanishes on all non-adjacent sides Γ_j, where $j \notin \{i-1, i, i+1\}$. We propose the rational function

$$B_i(d_1,\dots,d_n) = \frac{D_{i,i-1} + D_{i+1,i}}{\sum_j D_{j,j-1}}, \quad D_{i_1\dots i_k} = \prod_{j \notin \{i_1\dots i_k\}} d_j^2.$$

Due to the squared terms, the related partial derivatives of the blending functions vanish, i.e.,

$$\frac{\partial}{\partial d_k} B_i(d_1,\dots,d_j = 0,\dots,d_n) = 0$$

for $j \notin \{i-1, i+1\}$, $k \in [1\dots n]$.

2.2. Ribbon surfaces

Curved ribbons comprise the positional and tangential information along the boundaries. In contrast to linear ribbons, they deviate "moderately" from the transfinite patch to be created, and thus their combination produces a more predictable shape and less surface artifacts in strongly asymmetric curvenet configurations. Nevertheless, curved ribbons are composed of special linear ribbons and corner correction terms, as follows.

Let us assume that the tangential boundary information has already been specified by the user, or computed automatically based on the given curve network. Using these we can formulate conventional linear ribbons as $R_i(s_i,d_i) = P_i(s_i) + \gamma(d_i)T_i(s_i)$. In order to bring the 4-sided CR patch very close to the cubically blended Coons patch, we introduced a reparameterization function $\gamma(d_i) = \frac{d_i}{2d_i+1}$. Since $\gamma(0) = 0$ and $\gamma'(0) = 1$, it is easy to prove that the required interpolation properties are satisfied. We introduce a corner correction patch as

$$\begin{aligned} Q_{i,i-1}(s_i,s_{i-1}) &= P_i(0) + \gamma(1-s_{i-1})T_i(0) \\ &+ \gamma(s_i)T_{i-1}(1) + \gamma(s_i)\gamma(1-s_{i-1})W_{i,i-1}, \end{aligned}$$

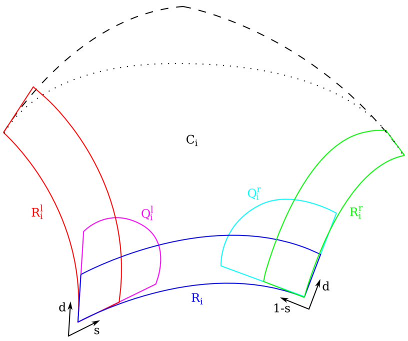
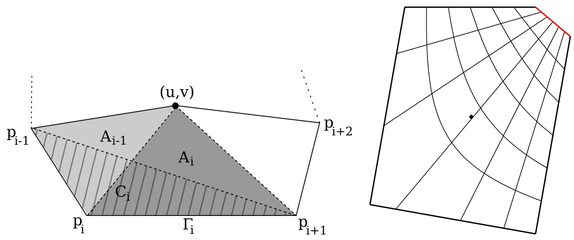


Figure 3: Construction of a curved ribbon.



(a) Construction **(b)** Isoparameter lines

Figure 4: Wachspress coordinate-based parameterization.

where $W_{i,i-1} = \frac{\partial}{\partial s_i} T_i(0) = -\frac{\partial}{\partial s_{i-1}} T_{i-1}(1)$ denotes the twist vector at the $(i, i-1)$-th corner (see [Far02]).

A *curved ribbon* is defined as the combination of three consecutive linear ribbons, and it is actually a *Coons patch with three of its four sides given,* defined over a local rectangular domain. Let $C_i(s_i, d_i)$ denote the curved ribbon for the i-th side; α_0 and α_1 are the cubic Hermite functions. We simplify the notation and drop the indices of s and d, as it does not cause any ambiguity. The definition of C_i is

$$\begin{aligned} C_i(s,d) &= R_i^l(s,d)\alpha_0(s) + R_i(s,d)\alpha_0(d) + R_i^r(s,d)\alpha_1(s) \\ &- \left[Q_i^l(s,d)\alpha_0(s)\alpha_0(d) + Q_i^r(s,d)\alpha_1(s)\alpha_0(d)\right], \end{aligned}$$

where $R_i^l(s,d)$, $R_i^r(s,d)$, $Q_i^l(s,d)$ and $Q_i^r(s,d)$ denote the ribbons and the correction patches on the left and right sides, respectively (see Fig. 3). We parameterize these by the local coordinates of the i-th side as follows:

$$R_i^l(s,d) = R_{i-1}(1-d, s), \qquad R_i^r(s,d) = R_{i+1}(d, 1-s),$$
$$Q_i^l(s,d) = Q_{i,i-1}(s, 1-d), \quad Q_i^r(s,d) = Q_{i+1,i}(d,s).$$

This construction constrains both s and d to lie in $[0,1]$.

2.3. Ribbon parameterization

The most crucial issue in all transfinite schemes is ribbon parameterization, i.e., how to compute the local side and distance parameters (s_i, d_i) from a given (u,v) domain point. This determines the associated points of the ribbons and thus has an essential effect on the shape. We have seen the requirement that $s_j, d_j \in [0,1]$ $(j \in [1 \dots n])$; it is also natural to require that each side parameter s_j is linear, and $d_i = 0$, $s_{i-1} = 1$, $s_{i+1} = 0$ are satisfied for all points lying on Γ_i. The distance parameters d_j $(j \in [1 \dots n])$ also change linearly along the sides, so on the i-th side $d_{i-1} = s_i$, $d_{i+1} = 1 - s_i$.

In the evaluation of parameterization methods there are two main issues: (i) the constant s_i, d_i parameter lines must have an even distribution in the domain, and (ii) the $(u,v) \to (s_i, d_i)$ mappings must be simple and computationally efficient. Let us deal with the s_i and d_i parameters separately.

In the so-called *linear sweep* parameterizations [VRS11], the $s_i =$ const. isolines are straight lines in the domain space; as s_i varies from 0 to 1 these lines sweep from side Γ_{i-1} to side Γ_{i+1}, for example using a linear mapping between them. As for the $d_i =$ const. isolines, applying Wachspress coordinates [Wac75] turned out to be a good solution, concerning shape and computational efficiency. Originally these assign weights to the corners of a polygon, but it is possible to compute distance isolines by them, as follows. The barycentric coordinates λ_i are defined as

$$\lambda_i(u,v) = w_i(u,v) / \sum_k w_k(u,v),$$

where the individual weights are computed by Figure 4a:

$$w_i(u,v) = C_i / (A_{i-1}(u,v) \cdot A_i(u,v)),$$

where $A_{i-1} = \triangle(p_{i-1}, (u,v), p_i)$, $A_i = \triangle(p_i, (u,v), p_{i+1})$ and $C_i = \triangle(p_{i-1}, p_i, p_{i+1})$ represent triangle areas [HF06].

Then distance d_i is computed as

$$d_i(u,v) = 1 - (\lambda_{i-1}(u,v) + \lambda_i(u,v)),$$

which satisfies the initial constraints and edge linearity, due to the properties of Wachspress coordinates. An example using the above construction of s_i and d_i is shown in Figure 4b.

2.4. Assembling the composite ribbon patch

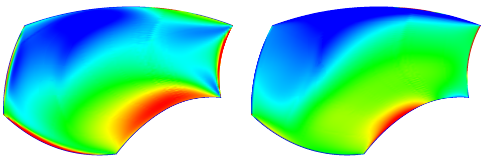
The CR patch has the simple formula:

$$S(u,v) = \frac{1}{2} \sum_{i=1}^{n} C_i(u,v) B_i(u,v).$$

According to the properties of the B_i blend functions, for any point on the i-th boundary all addends of the sum vanish except C_{i-1}, C_i and C_{i+1}. Since each of these ribbons also interpolates the corresponding three boundaries, the related three points on these ribbons are the same. Their cumulative blend is

$$B_{i-1} + B_i + B_{i+1} = (B_{i-1} + B_{i+1}) + B_i = 1 + 1 = 2,$$

which explains the division by two in the surface equation.



(a) Conventional patch **(b)** Composite ribbon patch

Figure 5: Mean maps for two different surface types.

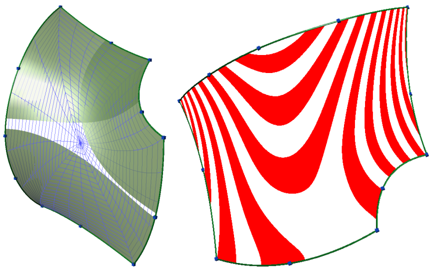


Figure 6: Ribbons and contours of a 5-sided patch .

CR patches ensure either parametric cross-derivative continuity (C^1), or match the tangent planes of the ribbons along the boundary (G^1). This depends on the ribbon parameterization. Wachspress parameterization provides G^1 continuity, which is sufficient for most surfacing applications. For details and related proofs see [Sal12].

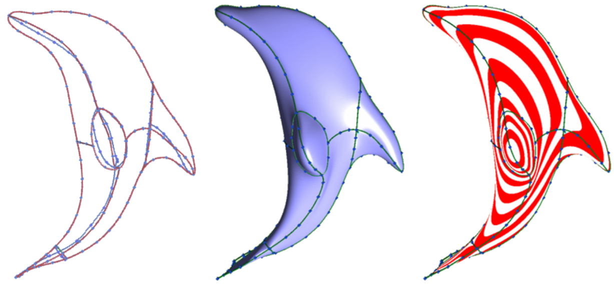
3. Examples

Former side-based transfinite schemes combined linear ribbons and applied different blending functions, see for example [Kat91]. While these patches are computationally simple, they may produce uneven curvatures in the vicinity of boundaries, due to the applied blending functions, that are singular at the corners (see also [VRS11]). The main motivation to develop our new schemes was to avoid these artifacts, see Figure 5. Figure 6 shows a patch with three of its curved ribbons, a spider-like net of radial isolines and contours.

The curve network in Figure 7 comes from a 3D drafting system (courtesy of Cindy Grimm [GJ12]). The network was interpolated by CR patches.

Conclusion

We have focused on the most crucial part of curve network-based design, i.e., how to represent collections of multi-sided transfinite surface patches that naturally fit onto general topology networks, *and* make shape editing easy and predictable. The proposed CR patch is a combination of curved ribbons and satisfy G^1 continuity. Challenging future research topics include fairing operations for curve network-based models and approximating polygonal meshes by transfinite patches.



(a) Curve network **(b)** Shading **(c)** Contouring

Figure 7: Dolphin test model.

Acknowledgments

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