A Multi-sided Bézier Patch with a Simple Control Structure

Tamás Várady†, Péter Salvi†, György Karikó‡

†Budapest University of Technology and Economics
‡ShapEx Ltd.

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Outline

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Surface Modeling

- Transfinite patches
  - E.g. Gregory patch, generalized Coons patch
  - Arbitrary number of sides
  - Depends only on the boundary
  - Little control in the interior

- Control network-based patches
  - E.g. B-spline, T-spline
  - (Mostly) four-sided
  - Detailed control in the interior
  - No default positions ⇒ not suited for hole filling
  - CAD-compatible

- Best of both worlds:
  - Multi-sided hole filling patch with control points
Other Multi-sided Surfaces with Control Networks

[LD89] Loop et al.  
*A multisided generalization of Bézier surfaces*  
⇒ S-patches have a very high number of control points

[War92] Warren  
*Creating multisided rational Bézier surfaces using base points*  
⇒ Based on Bézier triangles, max. 6 sides

[ZB97] Zheng et al.  
*Control point surfaces over non-four-sided areas*  
⇒ High-degree expression, max. 6 sides

[LS08] Loop et al.  
*G^2 tensor product surfaces over extraordinary vertices*  
⇒ Collection of quadrilaterals, only \( G^2 \) continuity inside

[SZ15] Sun et al.  
*G^1 continuity between toric surface patches*  
⇒ Lattice-based, not always symmetric
Curve by Corner & Side Constraints

- Reconstruction of the red curve by displacements
  - Green curve: only end constraints (cubic, position & tangent)
  - Blue curve: degree elevation $\Rightarrow$ new control point
  - Add a displacement curve (only one non-zero control point)

- Same principle works for surfaces (corners, sides, interior)
Number of Control Points

- Central control point even for odd degrees
- *n quadrants*, each quadrant has \(\left\lceil \frac{d+1}{2} \right\rceil \times l\) points
- \(d\): degree
- \(l = \left\lfloor d/2 \right\rfloor\): layers
- Total # of points: \(1 + n \times \left\lceil \frac{d+1}{2} \right\rceil \times l\)
- Comparable to conventional patches
Domain & Parameterization

Domain

- Convex polygonal domain on the \((u, v)\) plane
  - Regular polygons can be used
- Side-based local parameterization functions \(s_i\) and \(h_i\)
- Computed by (corner-based) Wachspress barycentric coordinates \(\lambda_i(u, v)\)

\[\begin{align*}
\Gamma_i & (u, v) \\
\Gamma_{i-1} & s_i(\lambda_{i-1}, \lambda_i) \\
\Gamma_{i+1} & h_i(\lambda_{i-1}, \lambda_i) \\
P_i & P_{i-2} \\
P_{i+1} & \end{align*}\]
Wachspress coordinates

Properties

- \( \lambda_i \geq 0 \) [positivity]
- \( \sum_{i=1}^{n} \lambda_i = 1 \) [partition of unity]
- \( \sum_{i=1}^{n} \lambda_i(u, v) \cdot P_i = (u, v) \) [reproduction]
- \( \lambda_i(P_j) = \delta_{ij} \) [Lagrange property]
Local Parameters

- $s_i = \frac{\lambda_i}{\lambda_{i-1} + \lambda_i}$
- $h_i = 1 - \lambda_{i-1} - \lambda_i$
- The above $s_i$ is undefined on “distant” sides
- Equivalent equation for $s_i$:
  \[ s_i = \frac{\sin(\theta_i)h_i}{\sin(\theta_i)h_{i-1} + \sin(\theta_{i-1})h_{i+1}} \]
- $h_j^\perp$ is the perpendicular distance from edge $\Gamma_j$ of the domain polygon
- $\theta_i$ is the angle at $P_i$
Bernstein Functions with Rational Weights

- $C_{j,k}^{d,i}$: $j$-th control point on side $i$, layer $k$
- Multiplied by $\mu_{j,k}^i B_j^d(s_i, h_i) = \mu_{j,k}^i B_j^d(s_i) B_k^d(h_i)$
- $\mu_{j,k}^i$ is rational in the corners
- $\alpha_i = h_{i-1}/(h_{i-1} + h_i)$, $\beta_i = h_{i+1}/(h_{i+1} + h_i)$
Central Weight & Patch Equation

- Weights do not add up to 1
- Deficiency $\Rightarrow$ weight of the central point:

$$B_0^d(u, v) = 1 - \sum_{i=1}^{n} \sum_{j=0}^{d} \sum_{k=0}^{l-1} \mu_{i,j,k} B_{j,k}^d(s_i, h_i)$$

- Patch equation:

$$S^d(u, v) = \sum_{i=1}^{n} \sum_{j=0}^{d} \sum_{k=0}^{l-1} C_{j,k}^{d,i} \mu_{j,k} B_{j,k}^d(s_i, h_i) + C_0^d B_0^d(u, v)$$
Interpolation Property

**Definition**

A Bézier ribbon is a Bézier patch given by the first two layers (rows) of control points on a given side.

**Theorem**

*The Generalized Bézier patch, on its boundary, interpolates the position and first cross-derivative of the Bézier ribbons of its respective sides.*
Degree Elevation & Reduction Algorithm

- Intuitive generalization of the algorithms for four-sided patches
- Central control point setting:
  - (a) Mass center of its adjacent control points
  - (b) Middle surface point held in the same position
- Retains boundary constraints
- But changes the interior
GB Patch from Mixed-degree Bézier Ribbons

1. Reduce ribbons to cubic
2. Create a cubic base patch
3. Elevate the GB patch
4. Add displacements on the sides to match the ribbons (of the same degree)
5. If the degree is less than the max. ribbon degree, go to step 3
Editing with Control Points & by Middle Surface Point

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A Complex Model—1

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A Multi-sided Bézier Patch with a Simple Control Structure
A Complex Model—II
A Complex Model—III

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A Multi-sided Bézier Patch with a Simple Control Structure
Summary

- Generalization of tensor product Bézier patches
- Suitable for hole filling:
  - Default positions for interior control points by degree elevation/reduction
  - Middle surface point can be set directly (fullness control)
- Also has some properties of conventional surfaces:
  - Simple & intuitive control net structure
  - Interpolates, at its boundary, Bézier patches with the same control points
- Good patch quality, $C^\infty$ continuity internally
Limitations & Future Work

- Limitations:
  - Not a convex combination for $n = 3$ (central weight is negative)
  - Ribbon elevation algorithm changes the surface interior
  - Exact derivative computation is complex

- Missing proofs:
  - Convergence to a limit surface with $d \to \infty$
  - Positivity of the central weight for $n > 3$ (numerically tested on a dense mesh)

- Future work:
  - Mesh approximation
  - Global fairing
Any Questions?

Thank you for your attention.

T. Várady, P. Salvi, Gy. Karikó
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