Title:
Intuitive Interior Control for Multi-Sided Patches with Arbitrary Boundaries

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Introduction:

The geometric modeling of many objects, ranging from simple household items to complex aesthetic designs, involves non-quadrilateral free-form surfaces. In CAD systems these are generally represented by one of the following methods (see also Fig. 1):

- *Trimming* takes a larger four-sided patch and trims it at the specified boundaries. The connection to adjacent surfaces will generally be inaccurate (even positionally), and inherent symmetries of the multi-sided surface may not be reproduced.
- *Splitting* divides the n-sided area into subpatches. The actual choice of subdivision affects surface quality, and maintaining G^1 or G^2 continuity along the subdividing curves as the patch is modified can be difficult.

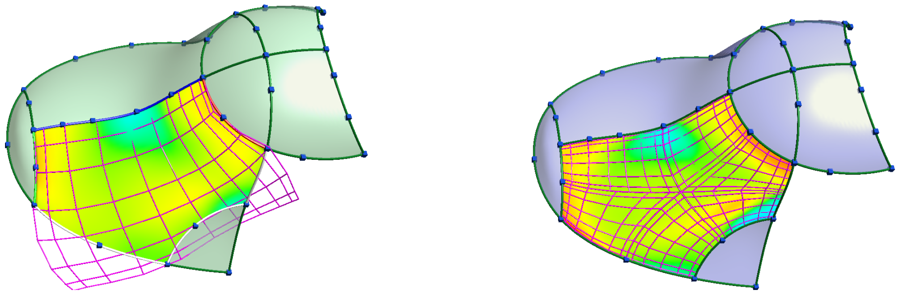

Fig. 1: Representing a 6-sided surface with trimming (left) and splitting (right).

In contrast, there are non-standard representations allowing an arbitrary number of sides without the drawbacks mentioned above. These are often called *transfinite interpolation* surfaces, as they reproduce the boundaries exactly, and can also ensure smooth connections to adjacent patches with G^1 or higher continuity.

Some of these can be regarded as multi-sided generalizations of the Coons patch, in the sense that the surface depends only on the positional and cross-derivative constraints at the boundaries. This is usually achieved by blending together ribbon surfaces interpolating some of the boundaries, so we will call these *ribbon-based* surfaces. Their strong point is also their weakness: since the geometry depends only on the boundaries, there is little control over the surface interior.

There are also *control-point–based* representations. These have fine-grained interior control, but are generally limited to (rational) polynomial boundaries.

The aim of this paper is to propose a genuinely multi-sided surface representation that (i) can handle any kind of boundary curves, (ii) allows connection to adjacent patches with G^1 or higher continuity, and (iii) has good control over the interior. In the following we will review some of the more influential ribbon- and control-point–based formulations, and show how these can be used to generate a patch satisfying all of the above requirements.

Preliminaries:

Almost all multi-sided surface formulations are defined over a 2D domain. Its shape can be varied, but often it is a regular n-sided polygon. Points inside the domain are then mapped to some sort of local coordinates depending on the actual patch equation. Here we show a set that can be described by generalized barycentric coordinates [1] and fits all schemes reviewed in this section.

Given a point in the domain and its generalized barycentric coordinates $\{\lambda_i\}$, we define

$$s_i(\lambda_1, \dots, \lambda_n) = \frac{\lambda_i}{\lambda_{i-1} + \lambda_i}, \qquad d_i(\lambda_1, \dots, \lambda_n) = 1 - (\lambda_{i-1} + \lambda_i) \tag{2.1}$$

with cyclic indexing. The side parameter s_i runs from 0 to 1 as we follow the i^{th} edge of the domain (connecting the $(i-1)^{\text{st}}$ and i^{th} vertices), giving 0 and 1 on the $(i-1)^{\text{st}}$ and $(i+1)^{\text{st}}$ edges, respectively. The distance parameter d_i vanishes at the i^{th} edge and increases monotonically, reaching 1 at the non-adjacent sides. Figure 2 shows constant parameter lines of $\frac{k}{10}$ $(k = 0 \dots 10)$. Note that s_i, $d_i \in [0, 1]$ and for a point on the i^{th} side $d_{i-1} = s_i = d_{i+1}$.

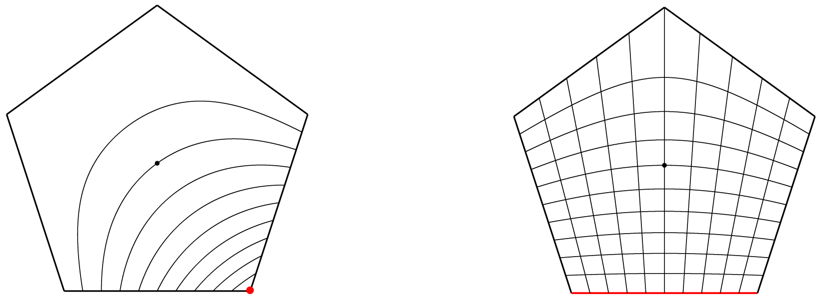

Fig. 2: Parameterization of a 5-sided domain, showing contours of Wachspress coordinates λ_i associated with the bottom-right corner (left) and the (s_i, d_i) system associated with the bottom side (right).

Ribbon-Based Surfaces. A ribbon $\mathbf{R}_i$ is a quadrilateral surface interpolating the i^{th} boundary curve, and also satisfying the associated cross-derivative constraints. We can blend different ribbons together using a variation [2] of Shepard's inverse distance weights (L_i):

$$\mathbf{S}(u, v) = \sum_{i=1}^{n} \mathbf{R}_i(s_i, d_i) L_i(d_1, \dots, d_n) = \sum_{i=1}^{n} \mathbf{R}_i(s_i, d_i) \cdot \frac{1/d_i^2}{\sum_{j=1}^{n} 1/d_j^2}. \tag{2.2}$$

Here (u, v) is a point in the 2D domain, s_i is a shorthand for $s_i(\lambda_1(u, v), \dots, \lambda_n(u, v))$ and similarly for d_i. This equation cannot be evaluated at the boundaries in this form, but this can be solved by multiplying

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both the numerator and denominator of L_i by $\prod_{k=1}^{n} d_k^2$. It is easy to see that the following properties hold for a point on the i^{th} boundary:

$$L_i = 1, \qquad L_j = 0 \quad (j \neq i), \qquad L'_k = 0 \quad (\forall k), \tag{2.3}$$

where the derivative is taken in an arbitrary parametric direction. A singularity remains, as the above equations specify both 0 and 1 for a corner point, but the limit of the sum exists. Increasing the exponents in L_i allows for connecting to adjacent surfaces with G^2 or higher continuity. Figure 3 shows a schematic depiction of this approach.

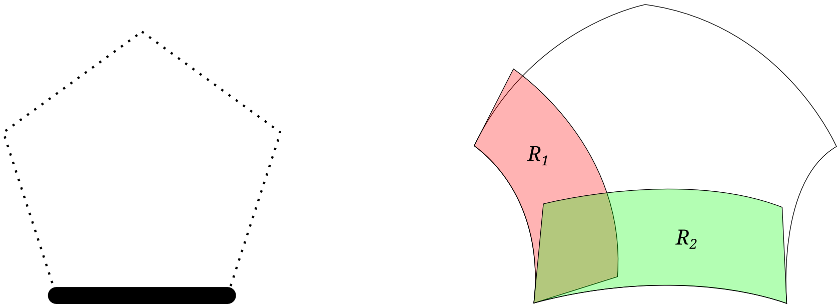

Fig. 3: Inverse distance weighted patch. Left: blending scheme (1 at one side and 0 on all others); Right: two linear ribbon surfaces.

Control-Point–Based Surfaces. A recent control-point–based surface representation is the Generalized Bézier or GB patch [3], which uses weighted Bernstein polynomials to satisfy the boundary constraints. Figure 4 shows its control structure and the weighting scheme.

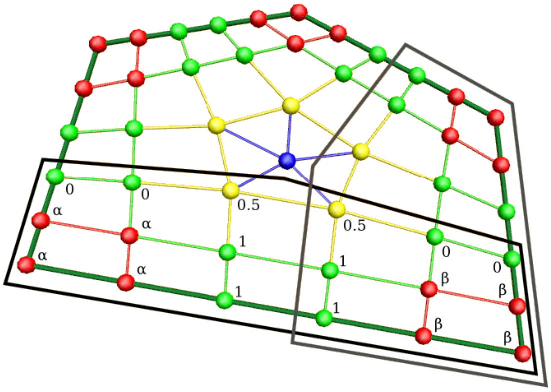

Fig. 4: Control net of the Generalized Bézier patch.

The black frames show which control points belong to the bottom and right-hand sides; some belong to both, and these are multiplied by both of the associated blending functions. The patch is defined as

$$\mathbf{S}(u,v) = \sum_{i=1}^{n} \sum_{j=0}^{p} \sum_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \mathbf{P}_{i,j,k} \cdot \mu_{i,j,k}(d_1,\ldots,d_n) B_j^p(s_i) B_k^p(d_i) + \mathbf{P}_0(1 - B_\Sigma(u,v)), \tag{2.4}$$

where p is the degree, $\mathbf{P}_{i,j,k}$ is the j^{th} control point in the k^{th} row associated with the i^{th} side, and B_i^p is the i^{th} Bernstein polynomial of degree p. The $\mu_{i,j,k}$ functions denote the weights shown in Fig. 4 with

$$\alpha_i = \frac{d_{i-1}^2}{d_{i-1}^2 + d_i^2}, \qquad \beta_i = \frac{d_{i+1}^2}{d_i^2 + d_{i+1}^2}. \tag{2.5}$$

The blends in the first term of Eq. (2.4) do not sum to 1; the second term associates the 'weight deficiency' with the central control point ($\mathbf{P}_0$), i.e.,

$$B_\Sigma(u,v) = \sum_{i=1}^{n} \sum_{j=0}^{p} \sum_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \mu_{i,j,k}(d_1,\ldots,d_n) B_j^p(s_i) B_k^p(d_i). \tag{2.6}$$

This patch behaves at its boundaries (in a G^1 sense) as a quadrilateral Bézier surface created with the associated control points. Increasing the exponents in Eq. (2.5) allows for higher order interpolation.

Hybrid Patch:

The proposed surface formulation takes Eq. (2.4) and replaces the outer two control rows with linear ribbons multiplied by the weight sum of the two rows, which is a singular blending function (L_i^*) similar to L_i in Eq. (2.2):

$$\mathbf{S}(u,v) = \sum_{i=1}^{n} \left[\sum_{j=0}^{p} \sum_{k=2}^{\lfloor \frac{p-1}{2} \rfloor} \mathbf{P}_{i,j,k} \cdot \mu_{i,j,k}(d_1,\ldots,d_n) B_j^p(s_i) B_k^p(d_i) + \mathbf{R}_i(s_i,d_i) L_i^*(u,v) \right] + \mathbf{P}_0(1 - B_\Sigma(u,v)), \tag{2.7}$$

with

$$L_i^*(u,v) = \sum_{j=0}^{p} \sum_{k=0}^{1} \mu_{i,j,k}(d_1,\ldots,d_n) B_j^p(s_i) B_k^p(d_i). \tag{2.8}$$

The main difference between L_i and L_i^* is that the latter does not sum to 1 in the interior of the domain, which leaves weight for the control points—the same weight they would have in a GB patch.

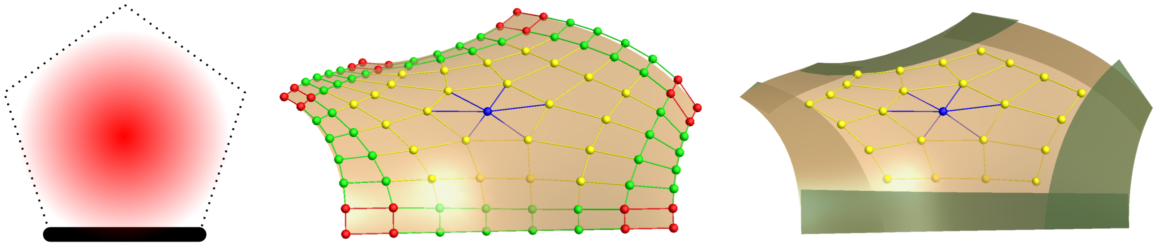

Fig. 5: Construction of the hybrid patch. Left: blending scheme; Middle: GB patch; Right: hybrid patch.

Replacing three control rows and increasing the exponent to 3 in Eq. (2.5) can interpolate the ribbons in a G^2 sense. In this case we need curved ribbons, which can be inherited from adjacent surfaces, or generated from a curve network.

The new patch representation is an improvement over the one in Eq. (2.2), as it adds interior control, and does not exhibit the large curvature variations that often appear near the boundaries with the steeply falling inverse distance weights. It is also more general than the GB patch, since it is not limited to polynomial boundaries; in fact, it can handle procedural boundary constraints, as well.

In Figure 6, a 5-sided patch with B-spline boundaries is modified by moving its control points (the ribbons remain unchanged), resulting in a much more natural isophote line distribution.

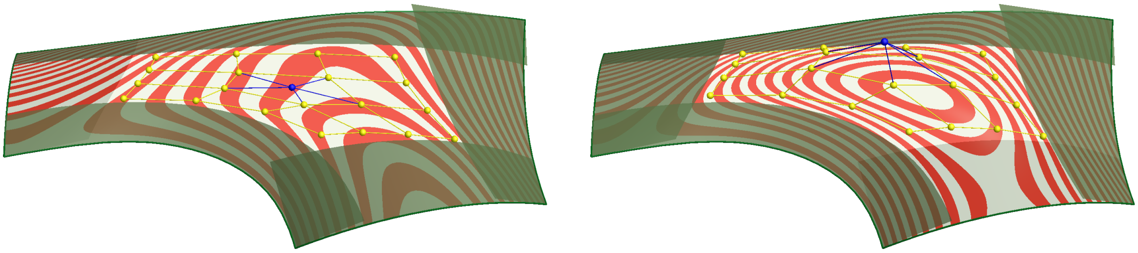

Fig. 6: Modification of a hybrid patch, showing isophote lines.

Figure 7 shows a five- and a six-sided hybrid patch. At the shared boundary a rotation-minimizing frame defines the normal fence both surfaces are perpendicular to.

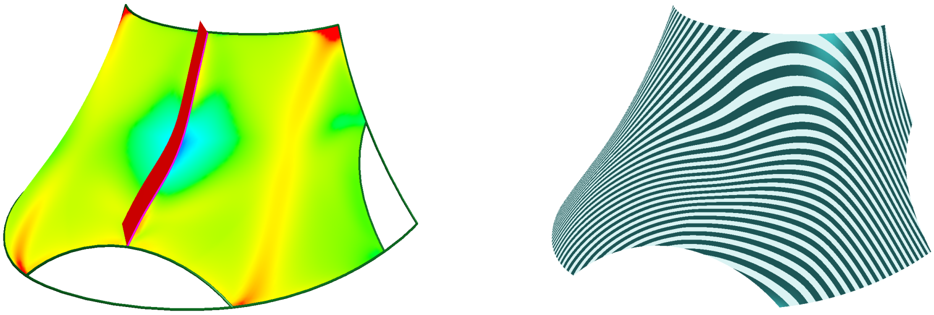

Fig. 7: Two patches connected with G^1 continuity; mean curvature with common normal fence (left) and contouring (right). Full red/blue colors are set to ± 0.015; the bounding box diagonal is 350 units long.

Conclusions:

We have proposed a new multi-sided surface formulation that combines the advantages of ribbon- and control-point–based patches. It can handle arbitrary boundary constraints, while also providing fine-grained control over the surface interior.

The full paper will include a review of other approaches to interior control, a discussion on limitations and the default positioning of control points, as well as more complex examples.

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