A Bird's-Eye View of Ribbon-Based Multi-Sided Surfaces

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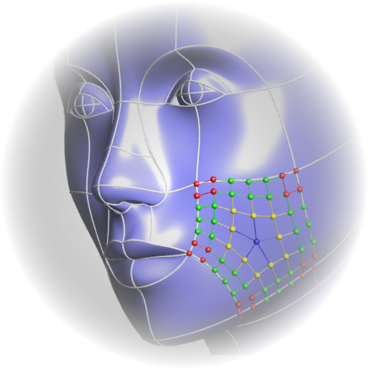
Budapest University of Technology and Economics

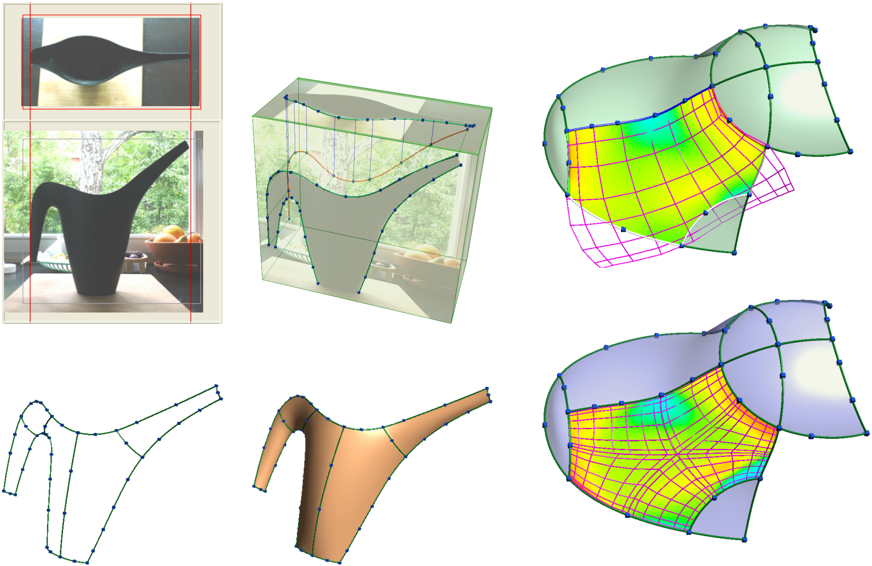
CAD Conference

July 1–3, 2024

Outline



Motivation



Survey

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Genuine multi-sided parametric surface patches – A survey

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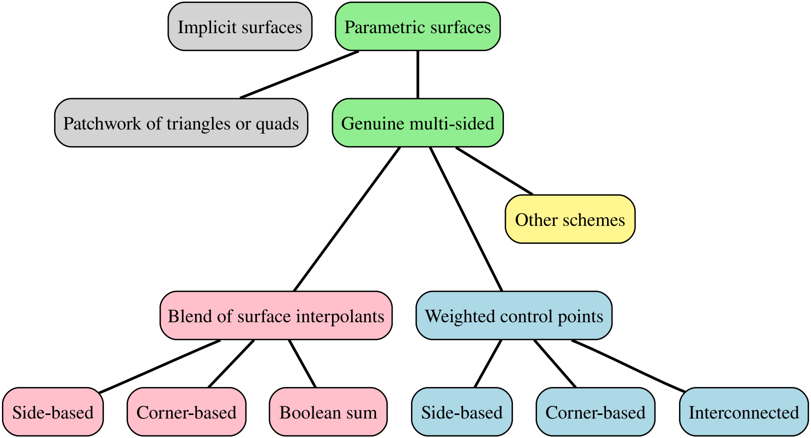
A R T I C L E   I N F O

Keywords:
Multi-sided surfaces
Ribbon-based surfaces
Transfinite interpolation
Control point patches
Surface modeling
General topology

A B S T R A C T

A state-of-the-art survey is presented on various formulations of multi-sided parametric surface patches, with a focus on methods that interpolate positional and cross-derivative information along boundaries.

Classification



Transfinite Interpolation

- Interpolate *infinitely many* points
- First use – Gordon & Hall (1973) ?
- General treatment of C^0 case in Sabin (1996)

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Mapping $\mathcal{M}$ between

$$\begin{aligned} \mathbf{B}(t) : \mathbb{R} \to \mathbb{R}^3 & \quad \text{parametric 3D boundary} \\ \mathbf{S}(u, v) : \mathbb{R}^2 \to \mathbb{R}^3 & \quad \text{parametric surface} \end{aligned}$$

s.t. $\forall t \, \exists (u, v) : \mathbf{B}(t) = \mathbf{S}(u, v)$ [similarly for multiple loops]

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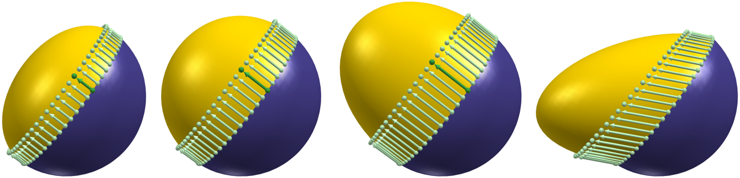
- Extension to C^1 in Sabin (1998)

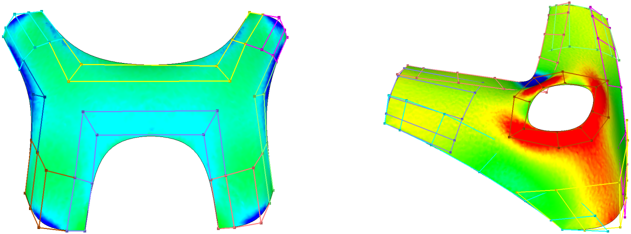
Integral-based ('elliptic') patches

Biharmonic (elliptic) PDE (e.g. Bloor & Wilson, 1989)

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)^2 \mathbf{S}(u, v) = \mathbf{0} \quad \text{(subject to boundary constraints)}$$



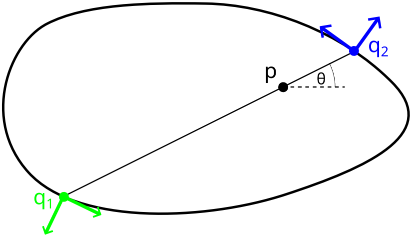
Jacobson et al. (2010)



Vaitkus (2023)

Integral-based ('elliptic') patches

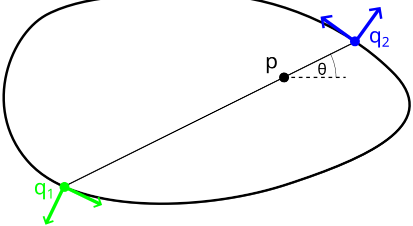
Gordon & Wixom (1974)



$$\mathbf{S}(\mathbf{p}) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{C}_\theta\left(\frac{\|\mathbf{p}-\mathbf{q}_1\|}{\|\mathbf{q}_2-\mathbf{q}_1\|}\right)\,\mathrm{d}\theta$$

Integral-based ('elliptic') patches

Gordon & Wixom (1974)



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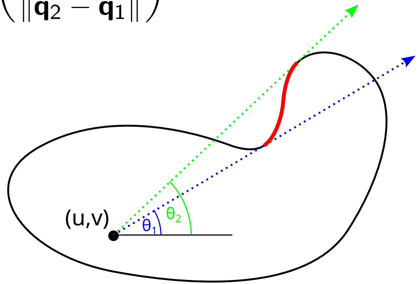
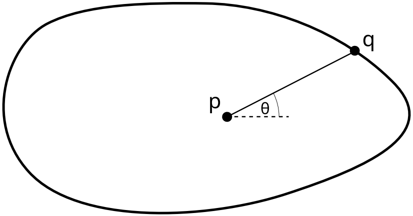
Extension to non-convex domains by Belyaev & Floater (2015)



Integral-based ('elliptic') patches

Transfinite mean value interpolation (Dyken & Floater, 2009)



$$\mathbf{S}(\mathbf{p}) = \int_0^{2\pi} \frac{\mathbf{S}(\mathbf{q})}{\|\mathbf{p}-\mathbf{q}\|}\,\mathrm{d}\theta \Big/ \int_0^{2\pi} \frac{1}{\|\mathbf{p}-\mathbf{q}\|}\,\mathrm{d}\theta$$

- Also for C^1
- Non-convex and multiply connected domains
- Explicit equations for special cases

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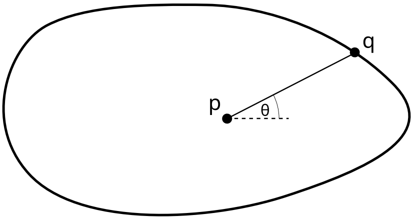
Transfinite mean value interpolation (Dyken & Floater, 2009)



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- Also for C^1
- Non-convex and multiply connected domains
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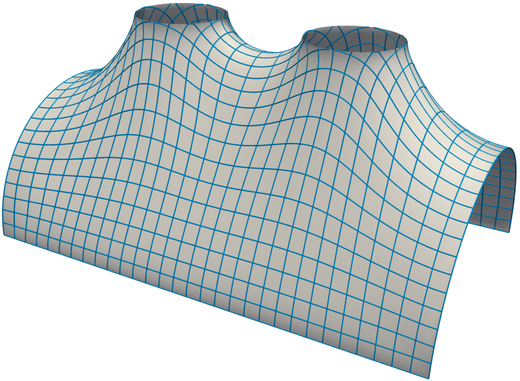
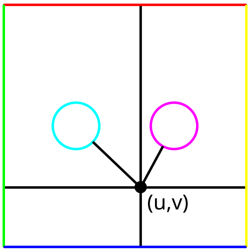
Pointwise radial minimization (Floater & Schulz, 2008)

- Hermite curves connecting **p** and **q**
- Find linear polynomial at **p** minimizing bending energy integral

Side-based ('hyperbolic') patches

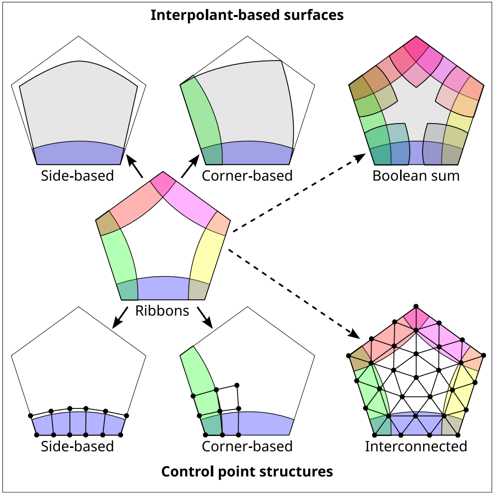
Sabin (1998)

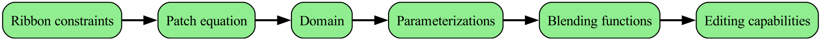




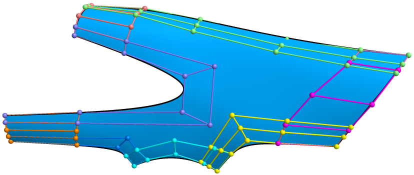
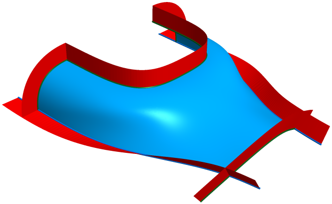
- Side-based version of 'pointwise radial minimization'
  - Unknown: 3D point + Jacobian
  - Minimize the elastic bend energy
- Interpolant-based surfaces ≈ 'hyperbolic'

Ribbon-based surfaces



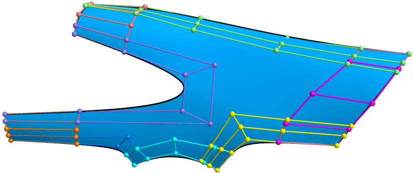


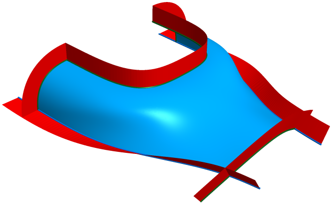
Ribbons

- Biparametric surface representing boundary constraints
  - Derived from a curve network (e.g. Chiyokura, 1986)
  - Geometric constraints (e.g. normal/curvature tensor field)
  - Control point grid
  - etc.

Ribbons

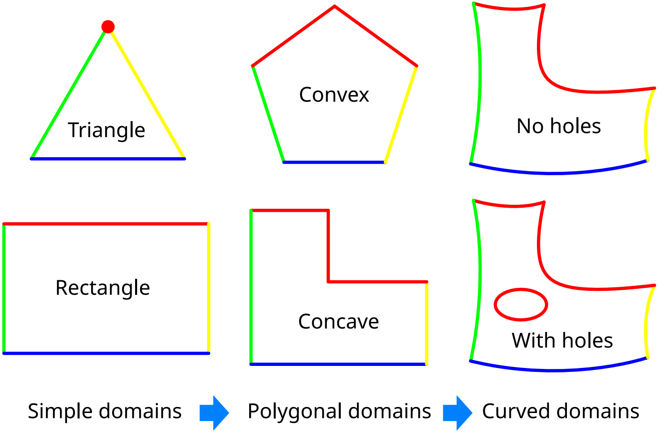




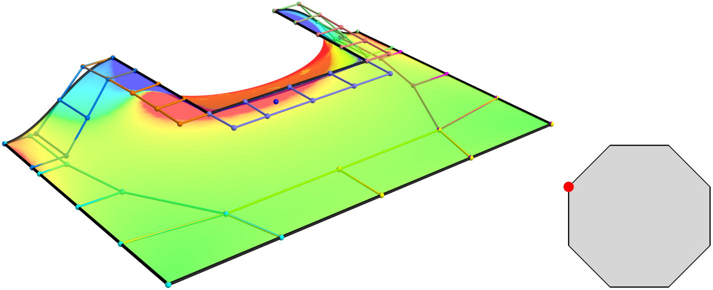
- Biparametric surface representing boundary constraints
  - Derived from a curve network (e.g. Chiyokura, 1986)
  - Geometric constraints (e.g. normal/curvature tensor field)
  - Control point grid
  - etc.
- Compatibility at the corners
  - Explicitly enforced
  - Rational blending – Gregory twist (Gregory, 1974)

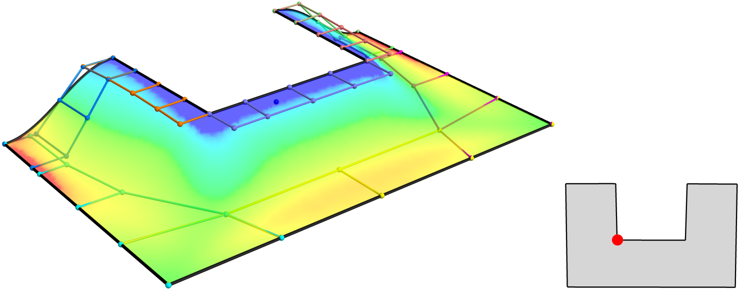
$$\mathbf{T}(u,v) = \frac{v}{u+v} \cdot \frac{\partial^2}{\partial u \partial v} \mathbf{R}_{i-1}(u,v) + \frac{u}{u+v} \cdot \frac{\partial^2}{\partial u \partial v} \mathbf{R}_i(u,v)$$

Domains

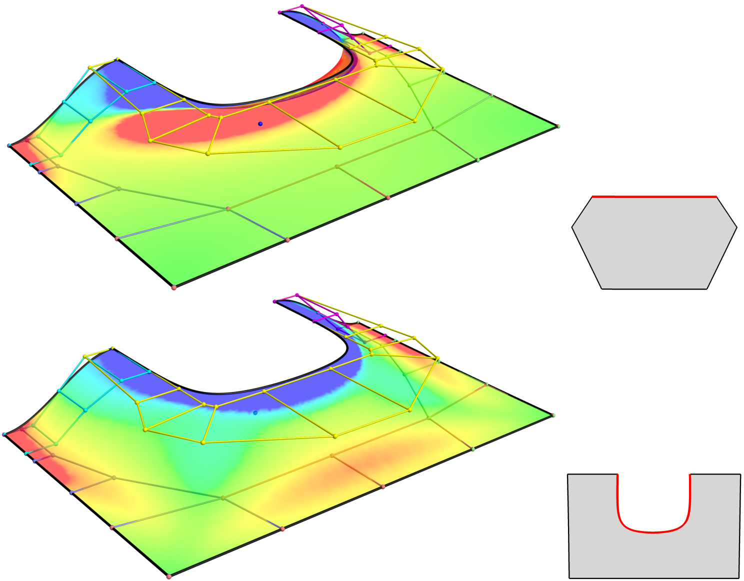

Triangle
Convex
No holes
Rectangle
Concave
With holes
Simple domains
Polygonal domains
Curved domains

Domains – Convex/Concave

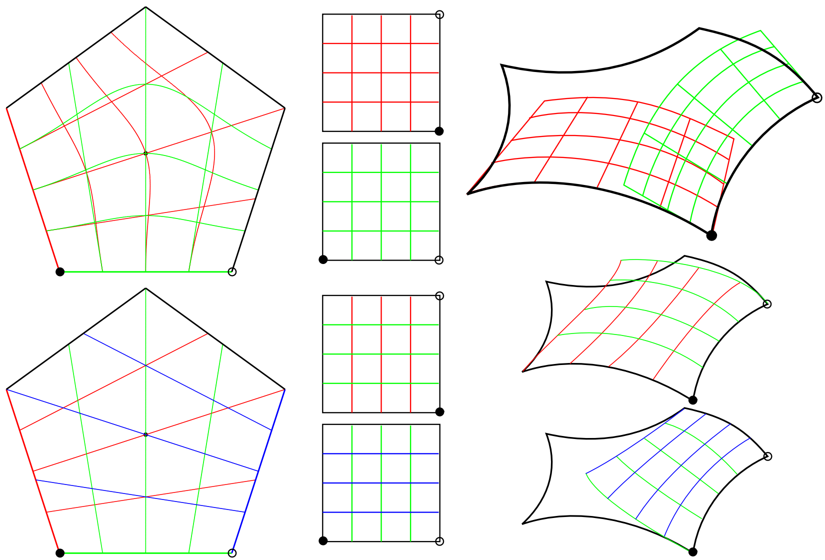




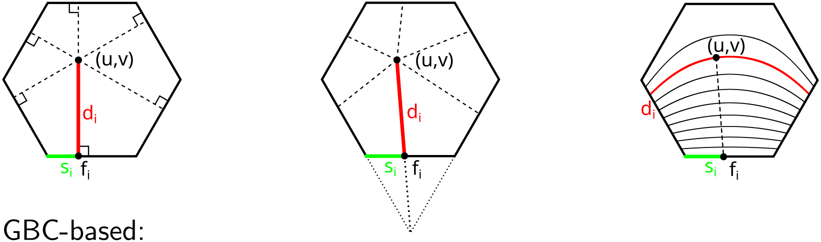
Domains – Polygonal/Curved



Local parameterizations

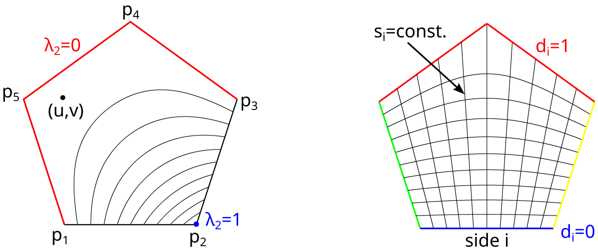


Parameterization constructions



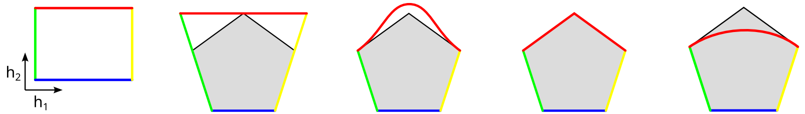
GBC-based:

$$d_i = 1 - \lambda_{i-1} - \lambda_i, \quad s_i = \lambda_i / (\lambda_{i-1} + \lambda_i)$$

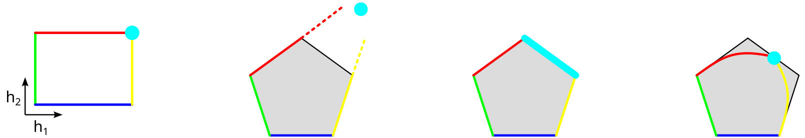


Parameterization types

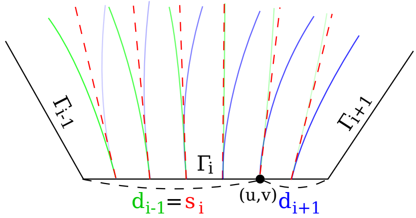
Side-based



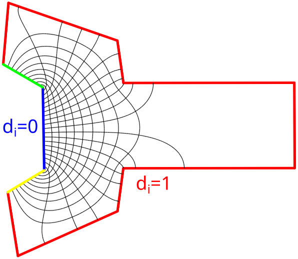
Corner-based

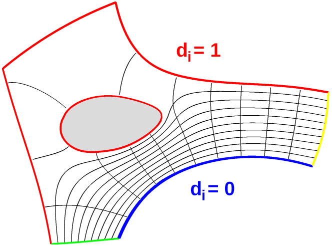


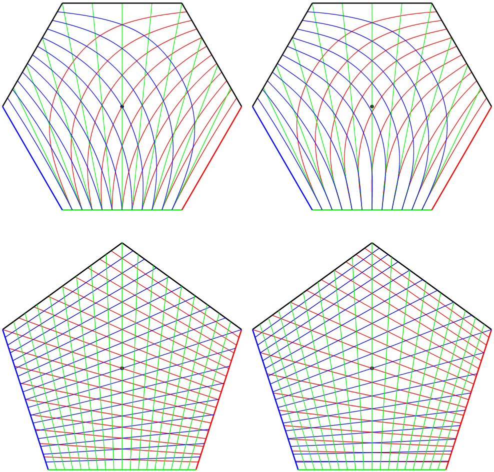
Derivative constraints



Examples







Outline

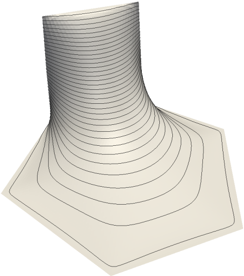
Singular side blend

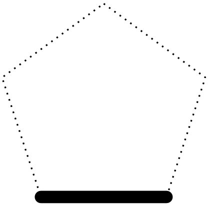
- Inverse distance weights (Shepard, 1968)
- Parametric distances (d_i)
- G^{e-1} interpolation

$$W_i^* = \frac{1/d_i^e}{\sum_{j=1}^n 1/d_j^e}$$

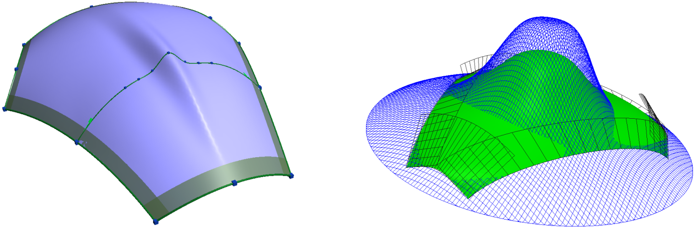
Patch equation (Barnhill, 1977; Gregory, 1986; Kato, 2000 etc.):

$$\mathbf{S}(u,v) = \sum_{i=1}^n \mathbf{S}_i^{\mathrm{Int}}(s_i, d_i) \cdot W_i^*(d_1, \ldots, d_n)$$

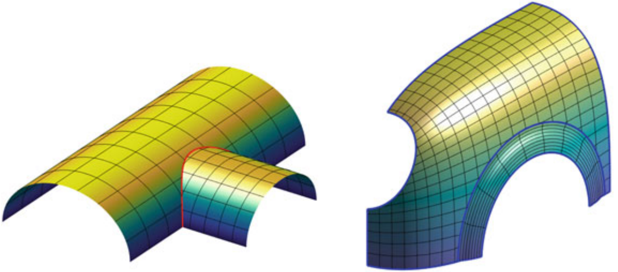




Singular side blend – Interior/boundary control



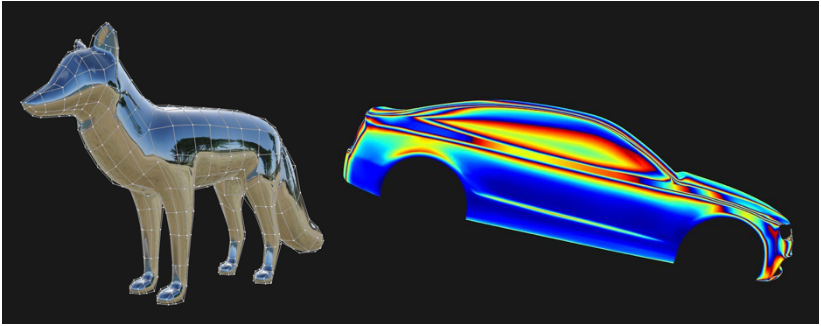
Várady et al. (2012)



Martin & Reif (2022)

Singular side blend – Polyhedral modeling

'SuperD' patch



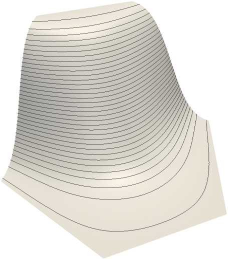
Rockwood & Gao (2018)

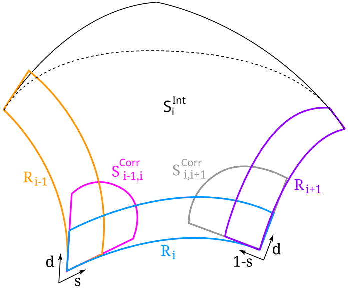
Non-singular side blend

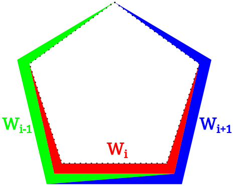
$$W_i = W_{i-1,i} + W_{i,i+1}, \text{ where } W_{i-1,i} = \frac{1/(d_{i-1}^2 d_i)^2}{\sum_{j=1}^{n} 1/(d_{j-1} d_j)^2}$$

Patch equation (Salvi et al., 2014):

$$\mathbf{S}(u,v) = \frac{1}{2} \sum_{i=1}^{n} \mathbf{S}_i^{\mathrm{Int}}(s_i, d_i) \cdot W_i(d_1, \ldots, d_n)$$





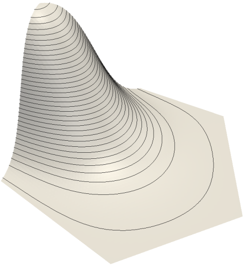


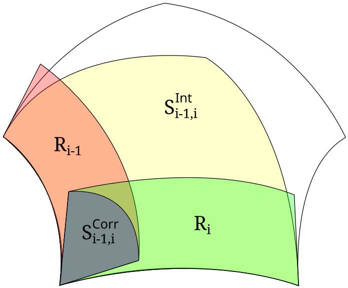
Corner blend

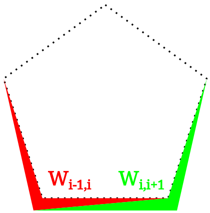
$$W_{i-1,i} = \frac{1/(d_{i-1}^2 d_i)^2}{\sum_{j=1}^{n} 1/(d_{j-1} d_j)^2}$$

Patch equation (Gregory, 1986):

$$\mathbf{S}(u, v) = \sum_{i=1}^{n} \mathbf{S}^{\mathrm{Int}}_{i-1,i}(s_{i-1}, s_i) \cdot W_{i-1,i}(d_1, \ldots, d_n)$$

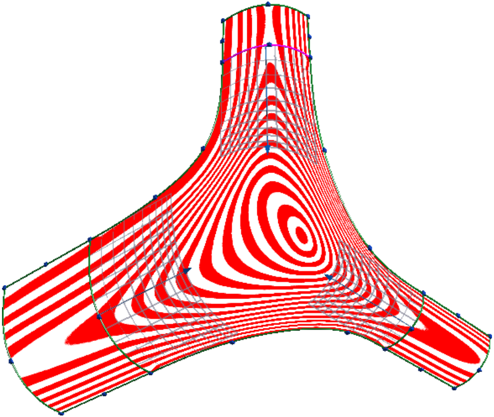






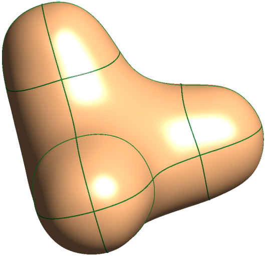
Corner blend – Extensions

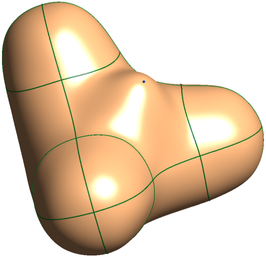
G^2 continuity



Salvi et al. (2014)

Midpoint control

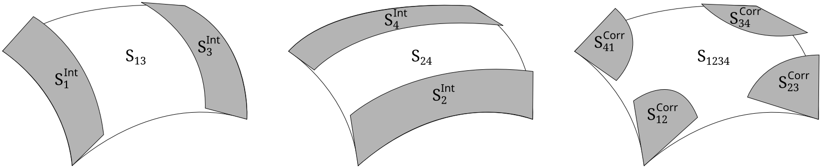




Salvi et al. (2016)

Boolean sum patch

Generalized Coons patch (Várady et al., 2011; Salvi et al., 2014)
(Classic Coons formula: $\mathbf{S} = \mathbf{S}_{13} \oplus \mathbf{S}_{24} = \mathbf{S}_{13} + \mathbf{S}_{24} - \mathbf{S}_{1234}$)



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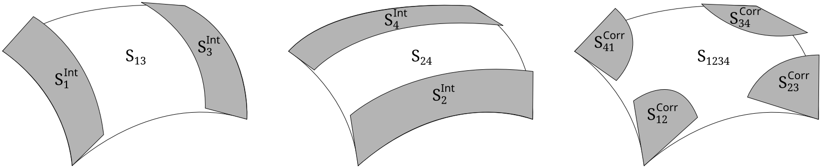
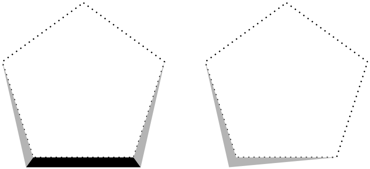


$$\mathbf{S}(u,v) = \sum_{i=1}^{n} \mathbf{S}_i^{\mathrm{Int}}(s_i, d_i) \cdot W_i(d_1, \dots, d_n)$$
$$- \sum_{i=1}^{n} \mathbf{S}_{i-1,i}^{\mathrm{Corr}}(s_{i-1}, s_i) \cdot W_{i-1,i}(d_1, \dots, d_n)$$

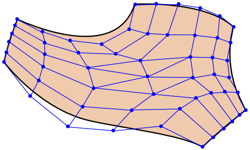


(needs constrained parameters)

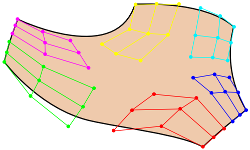
Outline

- Introduction
  - Motivation
  - Transfinite Interpolation
  - Basic Components
- Blend of Surface Interpolants
  - Side-based
  - Corner-based
  - Boolean sum
- Weighted Control Points
  - Interconnected
  - Corner-based
  - Side-based $\rightarrow$ new results
- Conclusion

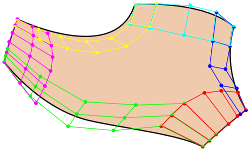
Control point configurations



interconnected

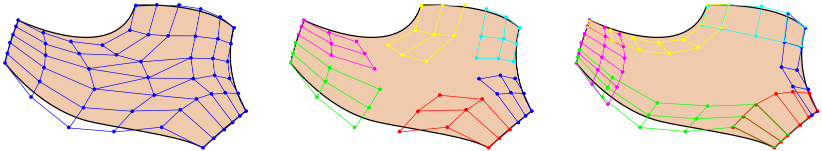


corner-based



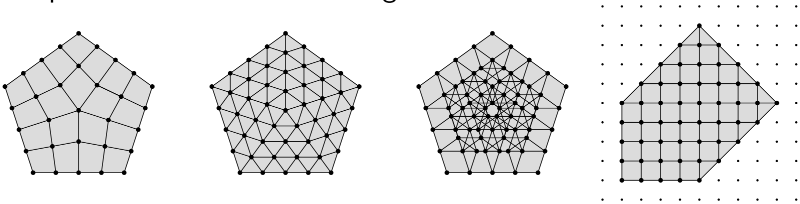
side-based

Control point configurations



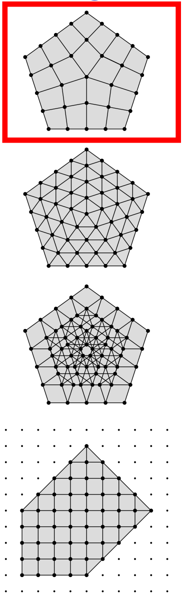
interconnected | corner-based | side-based

Examples of interconnected configurations:



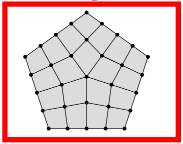
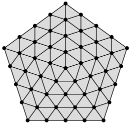
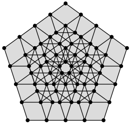
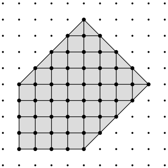
rectangular web | triangular web | Minkowski sum | lattice polygon

Rectangular spiderweb



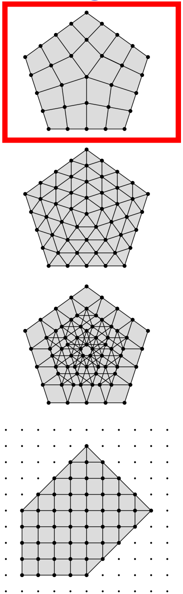
- Sabin (1983)
  - $n = 3, 5$
  - Quadratic

Rectangular spiderweb

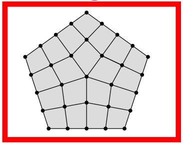
- Sabin (1983)
  - $n = 3, 5$
  - Quadratic
- Hosaka & Kimura (1984)
  - $n = 3, 5, 6$
  - Quadratic or cubic

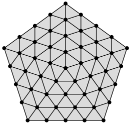
Rectangular spiderweb

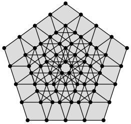


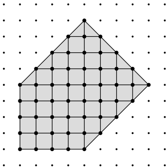
- Sabin (1983)
  - $n = 3, 5$
  - Quadratic
- Hosaka & Kimura (1984)
  - $n = 3, 5, 6$
  - Quadratic or cubic
- Zheng & Ball (1997)
  - $n = 3, 5, 6$ (and, in theory, others)
  - Any degree
- $\mathbf{S}(\mathbf{u}) = \sum_{\boldsymbol{i}} \mathbf{C}_{\boldsymbol{i}} B_{\boldsymbol{i}}(\mathbf{u})$
  - $\boldsymbol{i} = (i_1, \ldots, i_n)$ index vector (distances from each boundary)
  - $\mathbf{u} = (u_1, \ldots, u_n)$ parameters
  - $B_{\boldsymbol{i}}(\mathbf{u})$ blending function (quite complex)

Rectangular spiderweb









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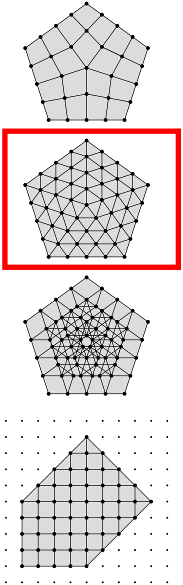
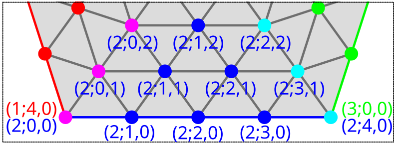
  - $\mathbf{u} = (u_1, \ldots, u_n)$ parameters
  - $B_{\boldsymbol{i}}(\mathbf{u})$ blending function (quite complex)
- Note: $\mathbf{S} : \mathbb{R}^n \supset \Omega \to \mathbb{R}^3$
- Ω is a *constrained* 2-dimensional subset of $\mathbb{R}^n$

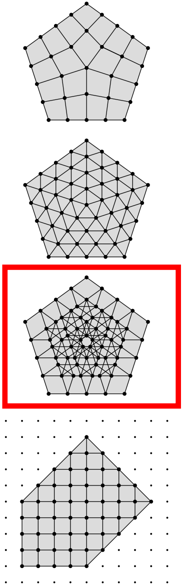
Triangular spiderweb



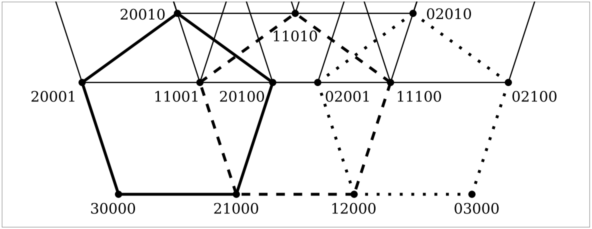
- M-patches (Karčiauskas, 1999, 2003)
- Based on GBCs $\lambda_1, \ldots, \lambda_n$
- $\mathbf{S}(u, v) = \sum_{\mathbf{I}_{n,d}} \mathbf{C}_{(i;j,k)} B^d_{(i;j,k)}(u, v)$
  - $\mathbf{I}_{n,d} = \{(i; j, k)\}$ index set
    - $1 \leq i \leq n$
    - $j, k \geq 0$
    - $j + k \leq d$
  - $B^d_{(i;j,k)}(u, v) = \binom{d}{j} \lambda_{i-1}^{d-j-k} \lambda_i^j \left(\prod_{\ell=1}^n \lambda_\ell\right)^k$ (can be more general)
  - Not a partition of unity → needs normalization
- Regular/convex polygonal domain



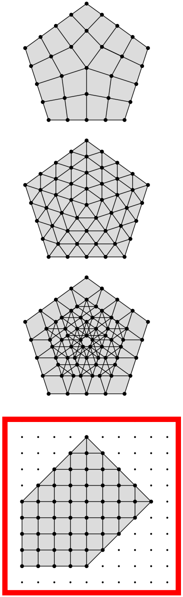
Minkowski sum



- S-patch (Loop & DeRose, 1989)
- Generalization of the Bézier curve/triangle
- Based on GBCs $\lambda_1, \ldots, \lambda_n$
- $\mathbf{S}(u,v) = \sum_{\mathbf{s} \in L_{n,d}} \mathbf{C_s} B_{\mathbf{s}}^d(u,v)$
  - $\mathbf{L}_{n,d} = \{\mathbf{s} = (s_1, \ldots, s_n)\}$ index set
    - $\forall i : s_i \geq 0$
    - $\sum_{i=1}^n s_i = d$
  - $B_{\mathbf{s}}^d(u,v) = \frac{d!}{\prod_{i=1}^n s_i!} \cdot \prod_{i=1}^n \lambda_i^{s_i}$
- Also on concave domains (Schaefer, 2017)

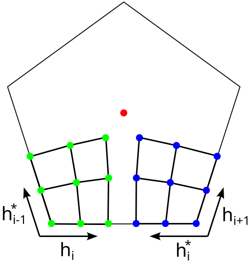


Lattice polygon



- Toric Bézier patch (Krasauskas, 2002)
- Inspired by *toric varieties* (algebraic geometry)
- Domain D = lattice polygon
- $\mathbf{S}(u, v) = \sum_{(j,k)\in D\cap\mathbb{Z}^2} \mathbf{C}_{(j,k)} B^d_{(j,k)}(u, v)$
  - $B^d_{(j,k)}(u, v) = c_{(j,k)} \cdot \prod_{i=1}^{n} h_i(u, v)^{h_i(j,k)}$
  - $c_{(j,k)}$ (almost) free coefficients
  - $h_i(u, v)$ perpendicular distance from side i (scaled s.t. gives integers at lattice points)
  - Not a partition of unity $\rightarrow$ needs normalization
- G^1/G^2 interpolation (Sun & Zhu, 2015, 2018)
- Degree elevation (Li et al., 2021)
- Not all configurations are admissible
- May be asymmetric
- Only convex domains

Corner-based constructions



Overlap patches (Várady, 1991; Salvi, 2022)

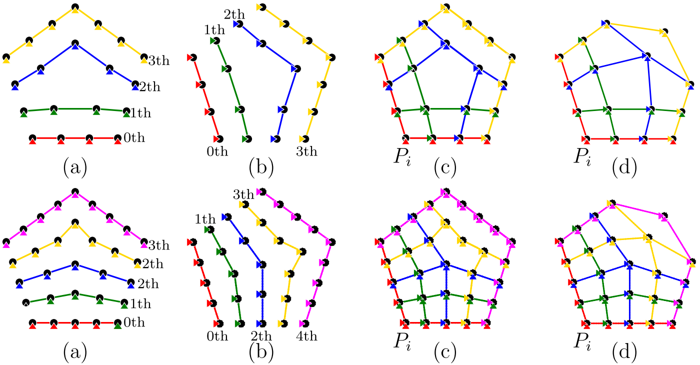
- Sum of quarter-Bézier patches
- $+$ central control point

- For an odd degree d:

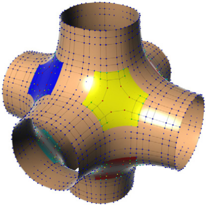
$$\mathbf{S}(u,v) = \sum_{i=1}^{n} \sum_{j=0}^{\lfloor d/2 \rfloor} \sum_{k=0}^{\lfloor d/2 \rfloor} \mathbf{C}_{j,k}^{i} B_j^d(h_i) B_k^d(h_{i-1}^*) + \mathbf{C}_0 B_0(u,v)$$

- $\mathbf{C}_{j,k}^{i}$: control point (j,k) in corner i
- $B_k^d(h)$: the k-th Bernstein polynomial of degree d
- $\mathbf{C}_0$: central control point
- $B_0(u,v)$: weight deficiency
- Needs a constrained parameterization

Corner-based constructions



Qin et al. (2023)



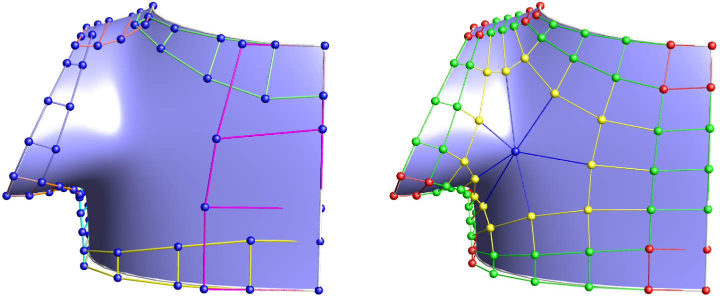
Blending Bézier patches (Qin et al., 2023)

- Corner ribbons from rectangular spiderweb
- Variant of the Charrot–Gregory scheme
- G^2 interpolation
- Needs a constrained parameterization

Side-based constructions

Generalized Bézier patches (Várady et al., 2016)

- Sum of half-Bézier patches + central control point

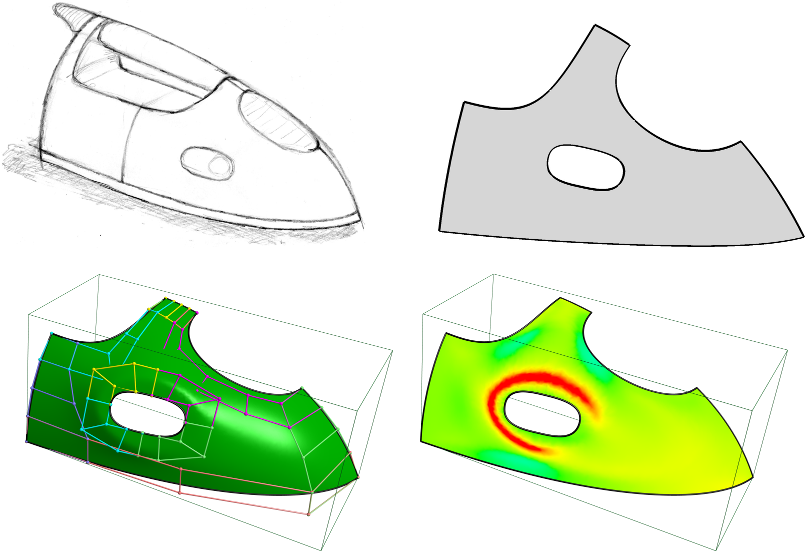


Várady et al. (2017)

$$\mathbf{S}(u,v) = \sum_{i=1}^{n}\sum_{j=0}^{d}\sum_{k=0}^{\lfloor\frac{d-1}{2}\rfloor} \cdot \mathbf{C}^i_{j,k}\underline{\mu_{i,j,k}(u,v)}B^{d,i}_{j,k}(u,v) \;+\; \mathbf{C}_0 \cdot \underbrace{B_0(u,v)}_{1-\sum\mu B}$$

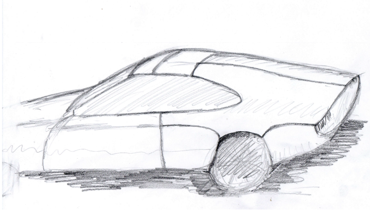
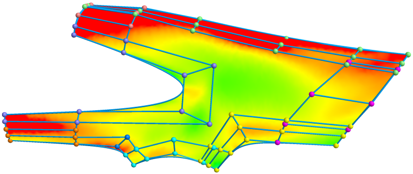
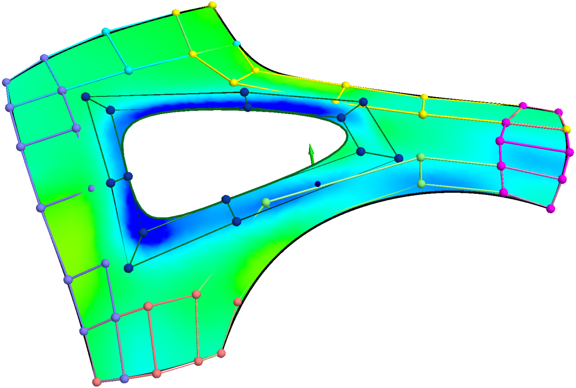
$$B^{d,i}_{j,k}(u,v) := B^d_j(s_i(u,v)) \cdot B^d_k(d_i(u,v))$$

Curved, multiply connected domains



Várady et al. (2020)

Generalized B-spline patches

Vaitkus et al. (2021)

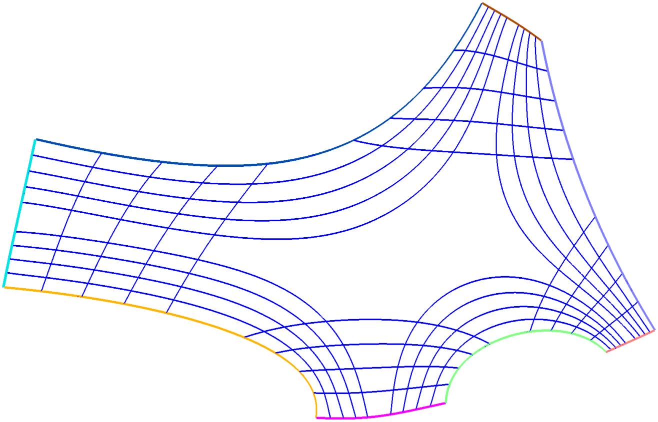
Outline

Interior control structure for Generalized Bézier patches

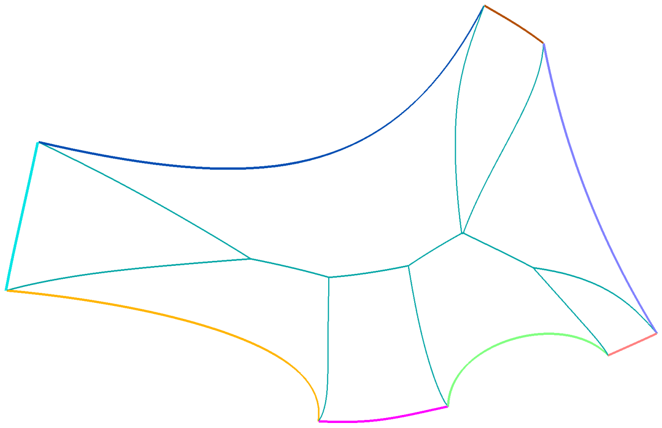
Problem

A single central control point may not be enough
in complex configurations.

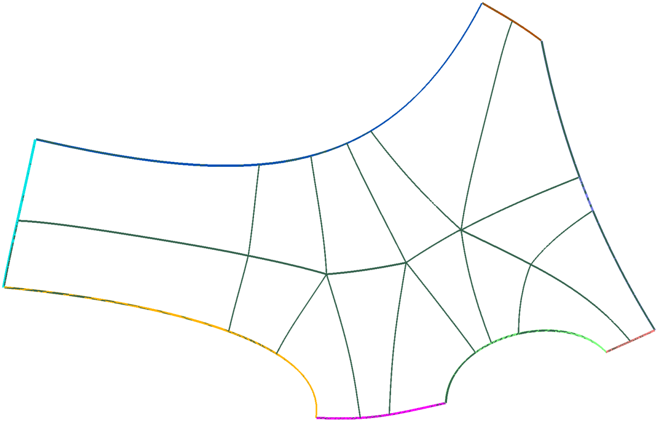
Distance parameters



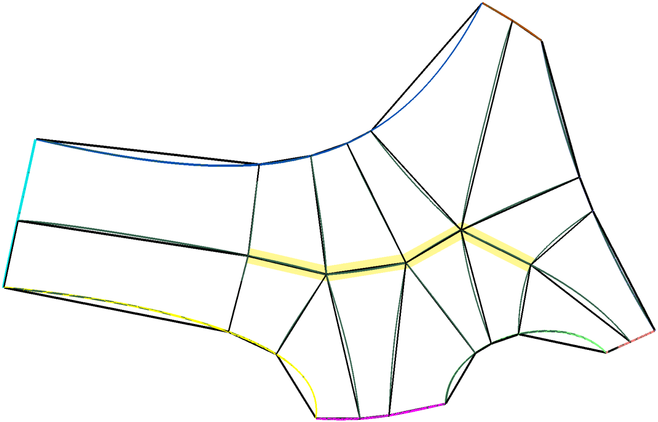
Parametric medial axis



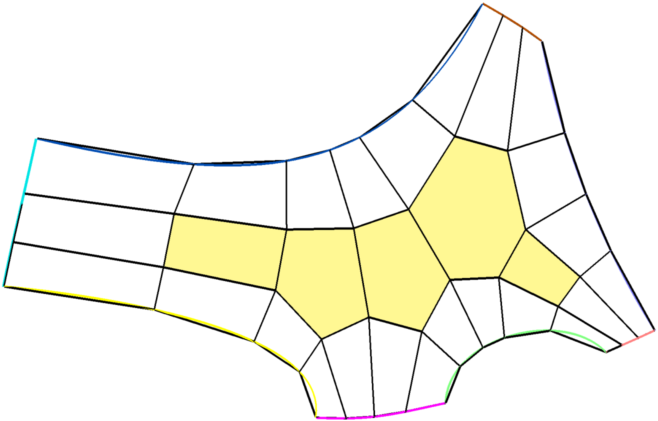
MAT-based quad structure



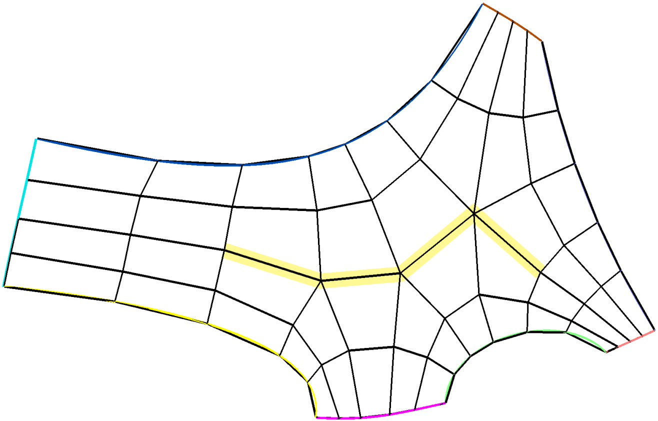
Depth-2 template



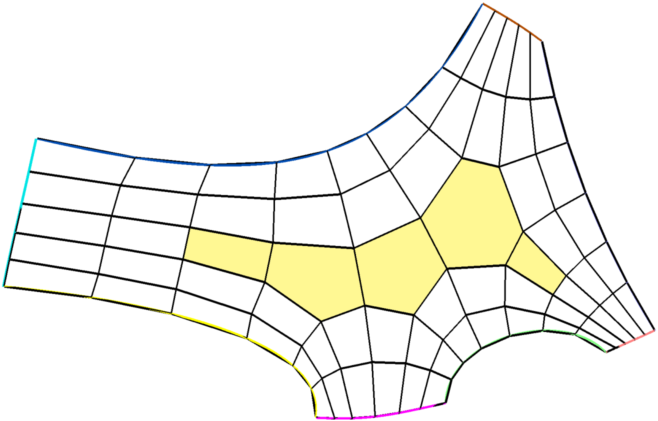
Depth-3 template



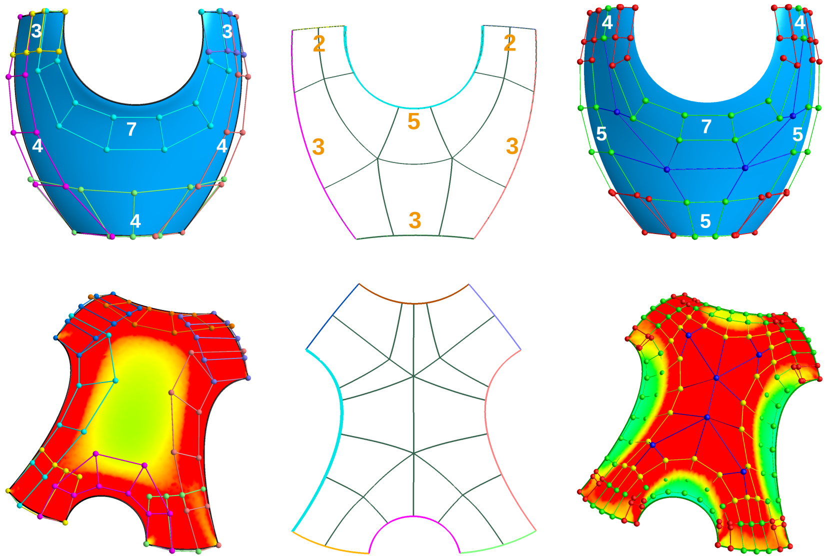
Depth-4 template



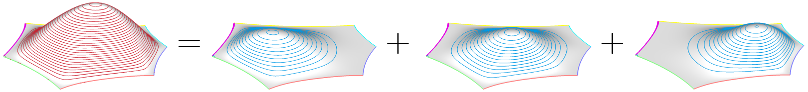
Depth-5 template

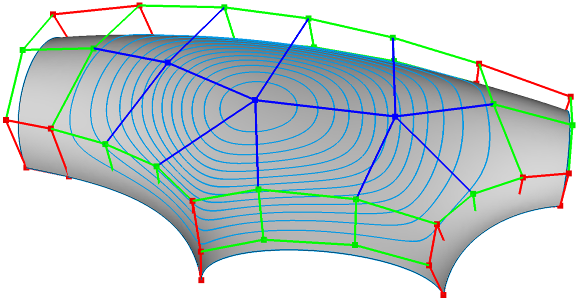


Degree synchronization



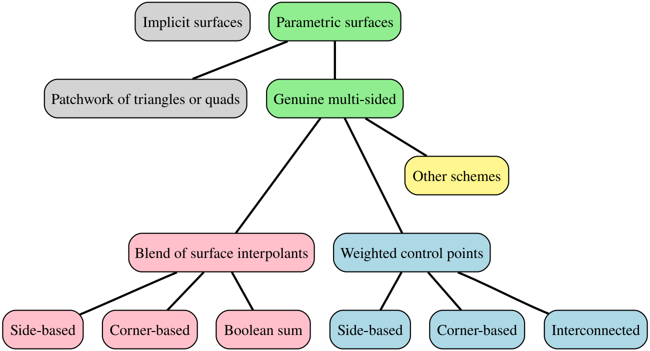
Distributing weight deficiency proportionally





Outline

Conclusion & future work



Surface type	Pros	Cons
Integral-based	Very general	Hard to evaluate
Interpolant-based	Arbitrary ribbons	No interior control
Control-net-based	Interior control	Polynomial boundaries

$\Rightarrow$ Generalized B-spline with interior control?

Related papers

1. Multi-sided patch survey
   T. Várady, P. Salvi, M. Vaitkus:
   Genuine multi-sided parametric surface patches – a survey.
   Computer Aided Geometric Design, Vol. 110, #102286, 2024.
2. Independent interior controls
   P. Salvi:
   Intuitive interior control for multi-sided patches with arbitrary boundaries.
   Computer-Aided Design and Applications, Vol. 21(1), pp. 143–154, 2024.
3. MAT-based interior controls
   M. Vaitkus, P. Salvi, T. Várady:
   Interior control structure for Generalized Bézier patches over curved domains.
   Computers and Graphics, Vol. 121, #103952, 2024.

https://3dgeo.iit.bme.hu/