Generalized Catenaries and Trig-Aesthetic Curves

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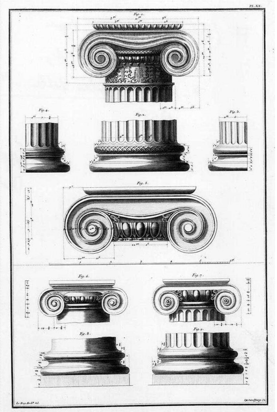
CAD'25

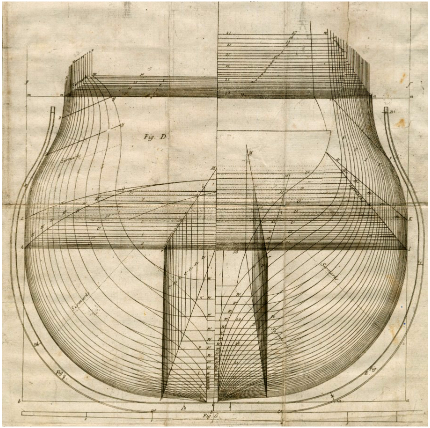
Shenzhen, June 23–25, 2025

Outline

MŰEGYETEM 1782

Classical Aesthetic Curves

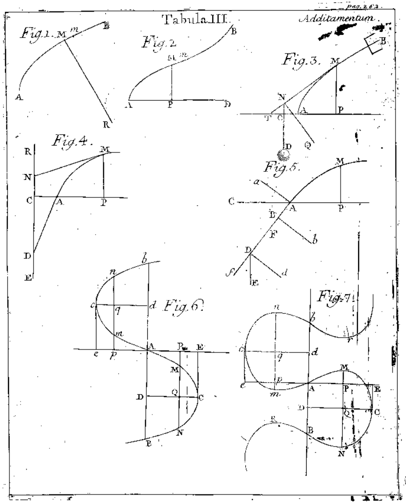


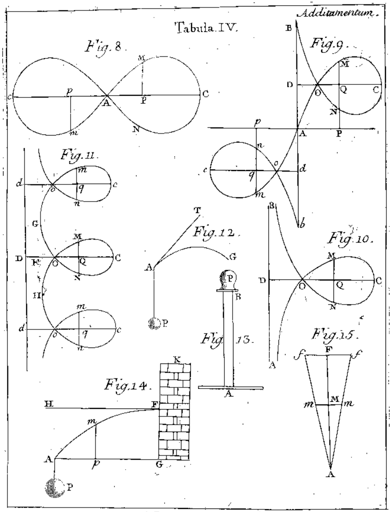


Spline energies [Hoschek–Lasser '96]

$$\kappa''(s) = 0 \qquad \text{(wooden)} \qquad \Rightarrow \text{clothoid}$$

$$\int \kappa(s)^2 \, \mathrm{d}s \to \min \qquad \text{(mechanical)} \qquad \Rightarrow \text{elastica}$$





Logarithmic Curvature Histogram (LCH)

Curve shape evaluation [Harada et al. '99]:

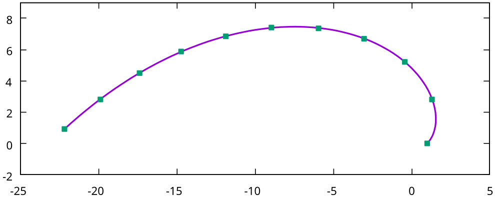
1. Take samples of the curvature radius (ρ_i) at equal arc lengths
2. Divide $\ln(\rho_i)$ into a fixed number of bins
3. Plot the logarithm of the percentage of samples in the bins

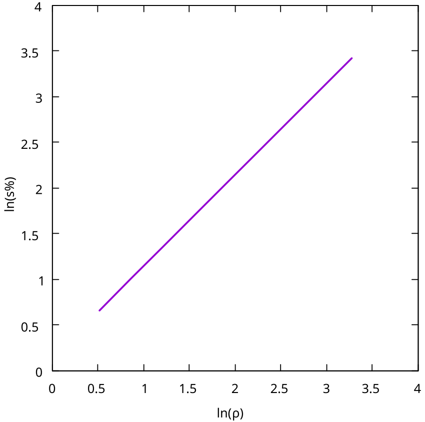
$\rightarrow$: $\ln \rho$

$$\uparrow : \ln \frac{\partial s}{\partial \ln \rho} = \ln \frac{\partial s}{\partial \rho / \rho}$$

Straight lines are favorable

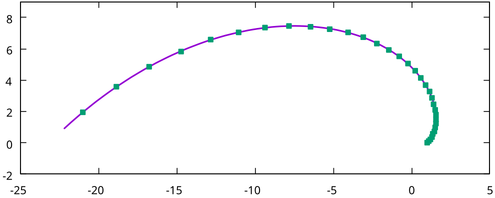
Logarithmic spiral

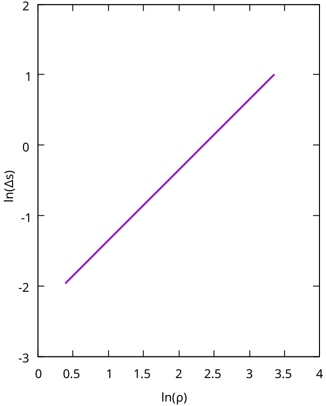




LCH—Alternative Interpretation

Logarithmic spiral





[Yoshida–Saito '06]:

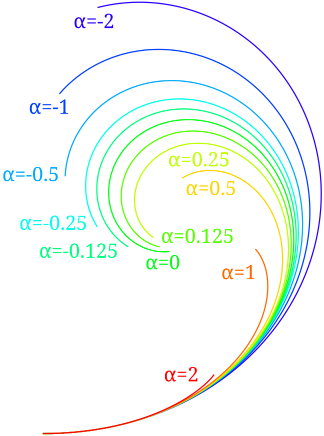
1. Divide the curve into segments with the same $\Delta\rho/\rho$ ratio
2. Draw the log–log plot of segment lengths, i.e., $\ln(\Delta s)$ over $\ln(\rho)$

Linearity means

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

where α is the slope

Log-Aesthetic Curves [Miura '06]

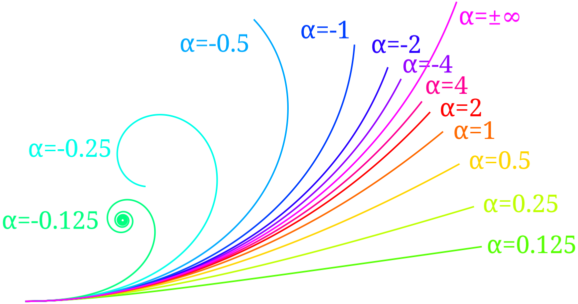


$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

$$\theta(s) = \frac{\alpha(c_0 s + c_1)^{(\alpha-1)/\alpha}}{(\alpha-1)c_0} + c_2$$

$$C(s) = P_0 + \int_0^s e^{i\theta(s)}\,\mathrm{d}s$$

$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s\right)$$



Standard form:

$$\mathbf{C}(0) = \mathbf{0}$$

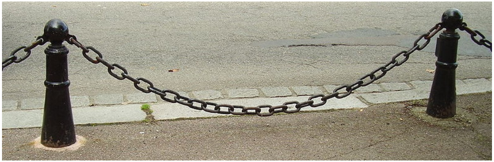
$$\theta(0) = 0$$

$$\kappa(0) = 1$$

$$\kappa'(0) = 1$$

Types of Log-Aesthetic Curves

- Circle ($c_0 = 0$)
- Circle involute ($\alpha = 2$)
- Logarithmic spiral ($\alpha = 1$)
  - $\theta(s) = \ln(c_0 s + c_1)/c_0 + c_2$
- Nielsen's spiral ($\alpha = 0$)
  - $\kappa(s) = \exp(c_0 s + c_1)$
  - $\theta(s) = \exp(c_0 s + c_1)/c_0 + c_2$
- Clothoid ($\alpha = -1$)



Generalized Catenaries

$$\kappa(s) = (c_0 s^2 + c_1 s + c_2)^{-1/\alpha}$$

- Generalization of LA curves (LA when $c_0 = 0$ or $c_1 = 2\sqrt{c_0 c_2}$)
- Includes catenaries: $\alpha = 1$, $c_0 = 1/a$, $c_1 = 0$, $c_2 = a$

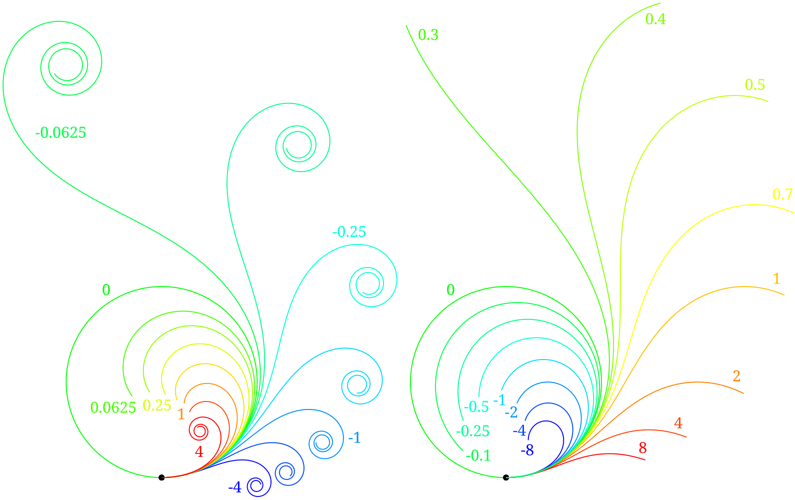
$$\kappa(s) = \frac{a}{s^2 + a^2}, \quad \theta(s) = \arctan(s/a) + c, \quad y = a\cosh(x/a)$$

- 'Hyperbolic–elastic' subfamily: $\alpha = -1$, $c_1 = 0$

$$\kappa(s) = c \cdot s^2 + 1, \quad \theta(s) = \frac{1}{3}c \cdot s^3 + s$$

  - $c > 0$: resembles hyperbolic spirals
  - $c < 0$: starts off similarly to elastica

Generalized Catenaries ($\alpha = -1$) vs. Elastica

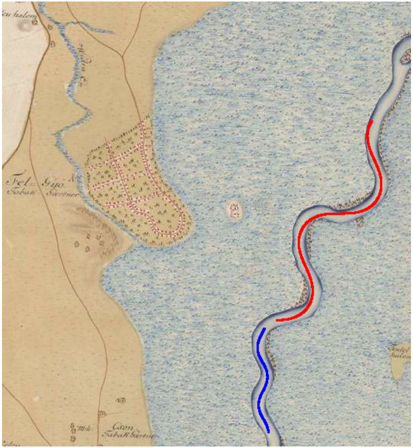


Trig-Aesthetic Curves

$$\kappa(s) = c_0 \cos(c_1 s + c_2), \quad \theta(s) = \frac{c_0}{c_1} \sin(c_1 s + c_2) + c_3$$

- 'Sine-generated curves'
- Used in geophysics (models river meandering)
- c_0: scaling
- c_1: **shape**
- c_2: starting parameter
- c_3: starting tangent
- Simpler version:

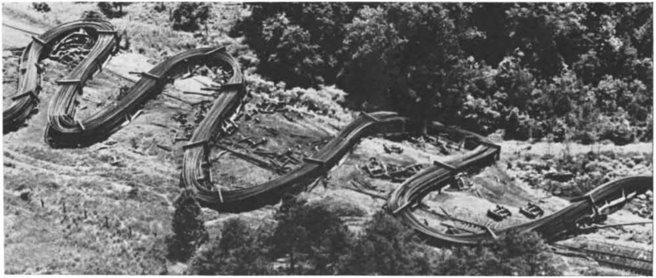
$$\kappa(s) = \cos(s/c)$$
$$\theta(s) = c \sin(s/c)$$



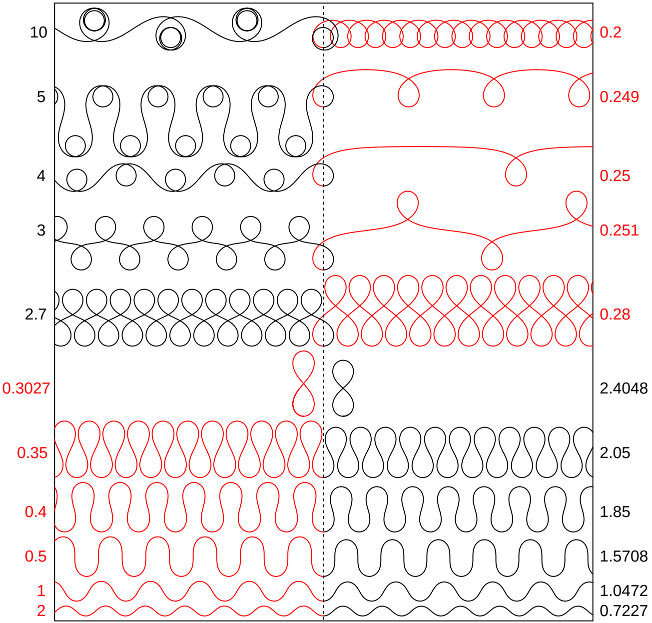
Connection with Elastica

$$\kappa(s) = \cos(s/c), \quad \theta(s) = c\sin(s/c)$$

- Rivers meander along elastic curves
  - Most probable path of a particle turning by normal distribution [von Schelling '51]
  - Minimize bending energy with fixed arc length
  - Solutions of $\theta''(s) + \lambda \sin\theta(s) = 0$
  - Maximum turning angle: $\arccos(1 - \frac{1}{2\lambda})$
- Sine-generated curves are similar [Langbein–Leopold '66]
  - Maximum turning angle: c



Trig-Aesthetic Curves vs. Elastica



Connection with Nielsen's Spiral

- Nielsen's Spiral (LA curve with $\alpha = 0$, $c_0 = 1/c$, $c_2 = 0$):

$$\theta_N(s) = c\exp(s/c + c_1),$$
$$\theta_N'(s) = \kappa(s) = \exp(s/c + c_1)$$

  Differential equation form:

$$\theta_N''(s) - \theta_N(s)/c^2 = 0$$

- Trig-aesthetic curve:

$$\theta(s) = c\sin(s/c), \quad \theta(s)'' = \kappa(s) = \cos(s/c)$$

  Differential equation form:

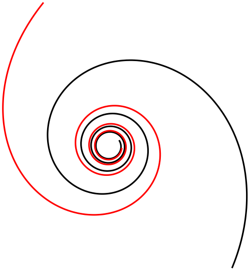
$$\theta''(s) + \theta(s)/c^2 = 0$$

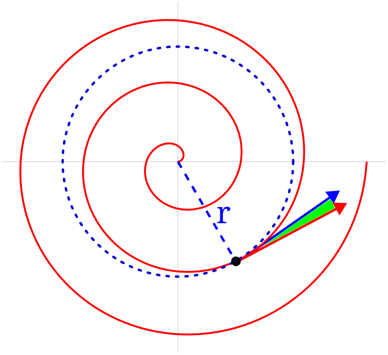
- Only the sign is different
  - Same when $c = -i$ (but initial values differ)

Connection with Hyperbolic Spiral ($c = -i$)

$$\kappa(s) = \cos(-s/i) = \cosh(s), \quad \theta(s) = \sinh(s)$$

- Pitch angle: angle between tangents to the spiral and a circle with the same center
- Hyperbolic spiral: pitch proportional to radius
- TA curve with $c = -i$: pitch converges to radius



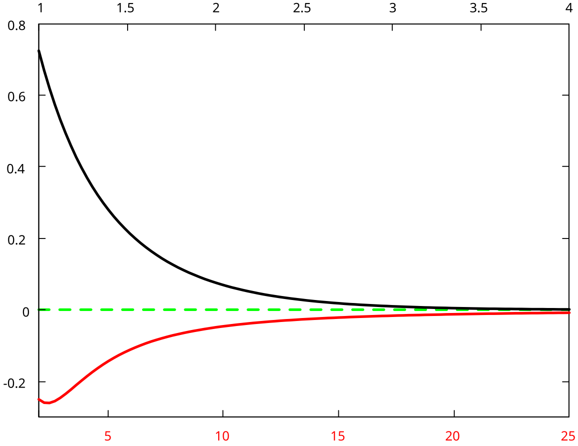


(Arithmetic spiral)

Connection with Hyperbolic Spiral ($c = -i$)

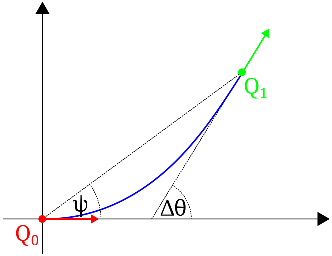
Comparison of the LCH slope function

$$\alpha(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right) = 1 - \frac{\rho(s)\rho''(s)}{\rho'(s)^2}$$



Hermite Interpolation

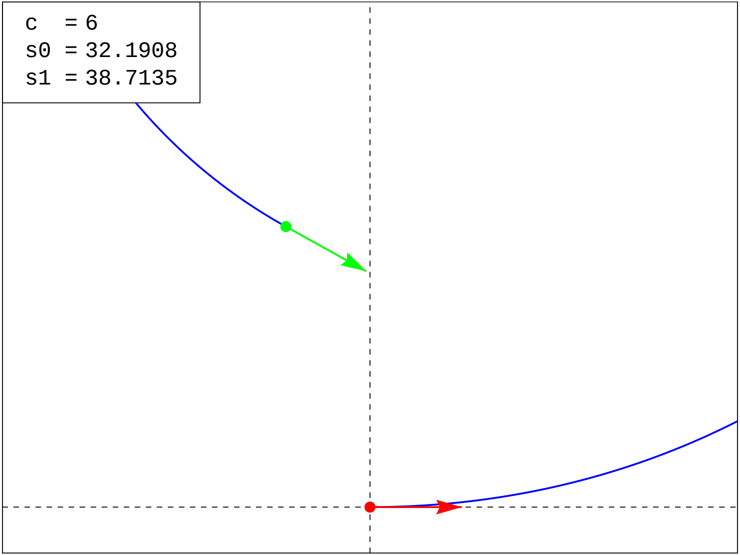
- Similarly to [Yoshida–Saito '06]
- Translation, rotation, scaling $\rightarrow$ irrelevant
- Simplified problem: two constraints (ψ and $\Delta\theta$)



- Variables: $[s_0, s_1]$ interval (c fixed)
- If we know $s_0 \Rightarrow$ we can compute s_1
- Determine s_0 by binary search
- Initial bracket by sampling

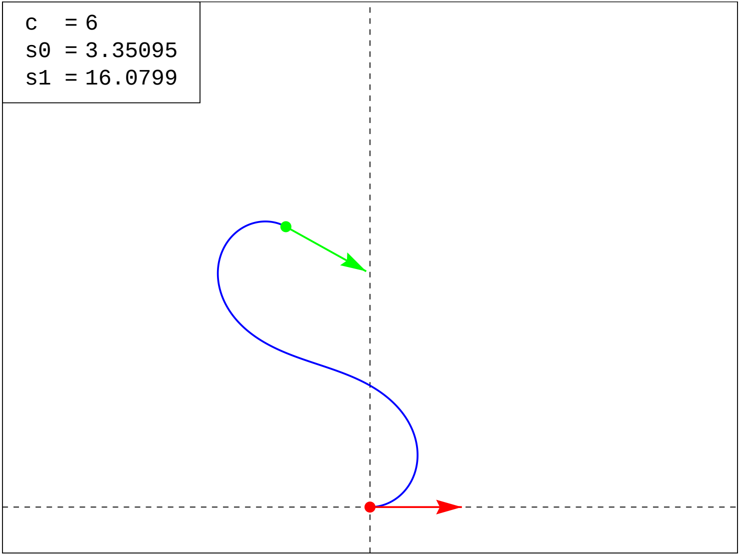
Hermite Interpolation—Choosing a Solution

Multiple solutions, some inferior



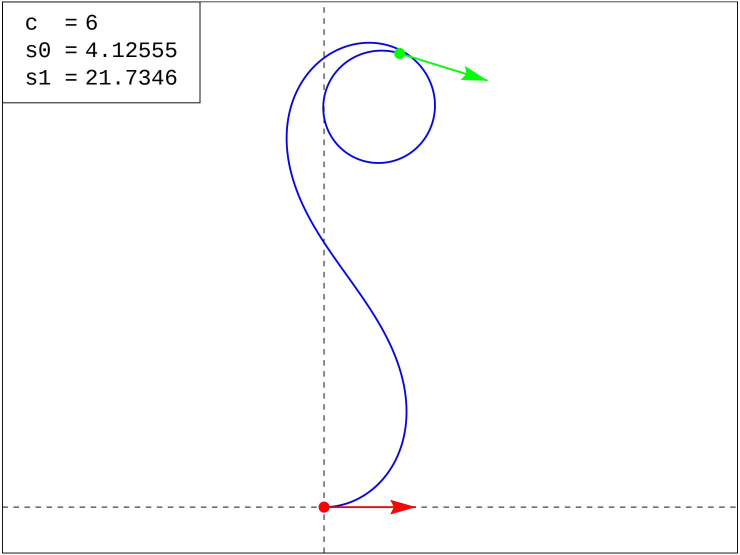
Hermite Interpolation—Choosing a Solution

Minimize arc length $E_s = \|\mathbf{Q}_1 - \mathbf{Q}_0\|(s_1 - s_0)/\|\mathbf{C}(s_1) - \mathbf{C}(s_0)\|$



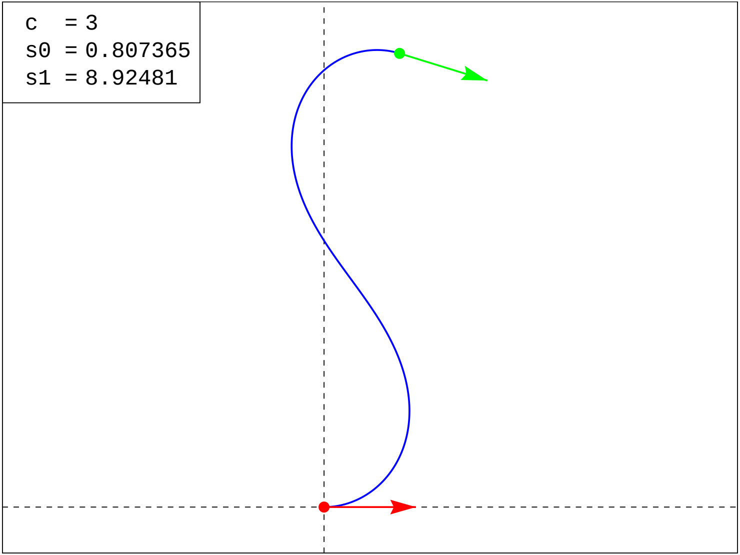
Hermite Interpolation—Choosing the Shape Parameter

Large c may result in loops



Hermite Interpolation—Choosing the Shape Parameter

Choose smaller c (but $c \geq |\Delta\theta|$)



Conclusion / Future work

- Generalized catenaries
  - A generalization of LA curves
  - Catenary curves
  - ‘Hyperbolic–elastic’ subfamily
- Trig-aesthetic curves
  - Approximates the elastica family
  - $\kappa(s)$ is a simple smooth function
  - Closely related to Nielsen’s spiral
  - With complex parameter approximates hyperbolic spirals
- Future work
  - Generalization to include Archimedean spirals?
  - $r = a + b\phi^{1/n}$
  - Arithmetic spiral, lituus, etc.

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