Generalized Catenaries and Trig-Aesthetic Curves

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Abstract. Two variations of the log-aesthetic curve family are presented: (i) a generalization that includes catenaries, and (ii) a trigonometric variant that approximates the elastica. Properties of these curves are investigated, and an algorithm for designing with the latter is also presented.

Keywords: fairness, log-aesthetic curves, elastica, sine-generated curve
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1 INTRODUCTION

Beautiful curves have fascinated mankind since its beginnings, as seen from prehistoric images of circles and spirals, and the ornamental designs of antiquity. Drawing them required special tools, and draftsmen—especially in shipbuilding—used wooden or metallic *splines*. Fixing some points with weights, the elasticity of these strips ensured a generally smooth shape. Hoschek–Lasser [6] (Sec. 3.7) gives the following equations for these in terms of the curvature as a function of arc length:

$$\kappa''(s) = 0 \quad \text{(wooden)}, \qquad \int \kappa(s)^2 \, \mathrm{d}s \to \min \quad \text{(mechanical)}. \tag{1}$$

The first equation means that curvature is a linear function, so it is a clothoid or Cornu spiral (also called an Euler spiral). The second leads to a differential equation studied thoroughly by (Jacob) Bernoulli and Euler, called the *elastica* [9].

The early work of Birkhoff [2] (p. 74) gives an intuitive definition for the *fairness* of curves:

> A first obvious requirement [..] is that the curvature varies gradually (that is, continuously) along the curve and oscillates as few times as possible in view of the prescribed characteristic points and tangents. In particular the curvature should not oscillate more than once on any arc of the contour not containing a point of inflection. [..] A second like requirement is that the maximum rate of change of curvature be as small as possible along the contour.

Figure 1: Left: Triskelion motif from the Neolithic Period (c. 3200 BC) in Newgrange, Ireland (image from Wikipedia). Right: Whale-shaped spline weights by Edson Marine.

In recent decades, the CAD community has tried to create curve representations that inherently possess these qualities. One salient example is the log-aesthetic curve of Miura [10]. This is a generalization of several classic aesthetic curve representations, including the circle and its involute, the logarithmic spiral, the Cornu spiral and Nielsen's spiral.

In this paper we will look at a generalization and a trigonometric variant of this curve family. The rest of the paper is structured as follows. First we review the log-aesthetic curve formulation, then we apply various modifications to the equation and analyze their effects. Finally we will show how to interpolate Hermite data with trig-aesthetic curves.

2 PRELIMINARIES

Log-aesthetic curves are based on the observation that the logarithmic curvature histogram (LCH) of fair curves tends to be a straight line [4]. Imagine that a curve is subdivided into small segments in such a way that the relative change in the radius of curvature along the segment ($\Delta\rho/\rho$) is constant [14]. Then the LCH is, in essence, the log–log plot of the lengths of these segments (Δs) plotted against ρ. But since $\Delta\rho/\rho$ is constant, we can plot $\Delta s/(\Delta\rho/\rho)$ instead of Δs. The linearity of this graph is formulated as

$$\frac{\mathrm{d}s}{\mathrm{d}\rho/\rho} = \hat{c}_0 \cdot \rho^{\alpha} \quad \equiv \quad \frac{\mathrm{d}s}{\mathrm{d}\rho} = \hat{c}_0 \cdot \rho^{\alpha-1}, \tag{2}$$

where $\hat{c}_0$, α are suitable constants and s is the arc length. Integrating by ρ, we arrive at

$$s = \frac{\hat{c}_0}{\alpha} \cdot \rho^{\alpha} + \hat{c}_1, \tag{3}$$

from which we get an equation for the radius of curvature:

$$\rho(s) = \left(\frac{(s-\hat{c}_1)\alpha}{\hat{c}_0}\right)^{1/\alpha}. \tag{4}$$

Introducing $c_0 = \alpha/\hat{c}_0$ and $c_1 = -\hat{c}_1 c_0$, a nice expression for the curvature is obtained:

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}. \tag{5}$$

In other words, a given power of the curvature, parameterized by arc length, should be linear.

We can also calculate the tangent angle by integrating the above formula by s:

$$\theta(s) = \frac{\alpha(c_0 s + c_1)^{(\alpha-1)/\alpha}}{(\alpha-1)c_0} + c_2. \tag{6}$$

With this, we can already plot the curve:

$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s \right), \tag{7}$$

or even more compactly as $C(s) = P_0 + \int_0^s e^{i\theta(s)}\,\mathrm{d}s$ in the complex plane.

Before continuing, let us note some special cases.

- *Circle*. When $c_0 = 0$ the curvature is constant, and $\theta(s) = c_1 s + c_2$.
- *Circle involute*. When $\alpha = 2$ the radius of curvature is proportional to $\sqrt{s}$.
- *Logarithmic spiral*. When $\alpha = 1$ the radius of curvature is proportional to the arc length; note that in this case

$$\theta(s) = \frac{\ln(c_0 s + c_1)}{c_0} + c_2. \tag{8}$$

- *Nielsen's spiral*. When $\alpha = 0$, the right-hand side of Eq. (2) becomes $\hat{c}_0/\rho$, so its integral is

$$s = \hat{c}_0 \cdot \ln\rho + \hat{c}_1, \tag{9}$$

  and with $c_0 = -1/\hat{c}_0$ and $c_1 = -\hat{c}_1 c_0$, the equations for the curvature and the tangent become

$$\kappa(s) = \exp(c_0 s + c_1), \qquad \theta(s) = \frac{\exp(c_0 s + c_1)}{c_0} + c_2. \tag{10}$$

- *Clothoid*. When $\alpha = -1$, curvature is linear in arc length.

Not all of the parameters (α, c_0, c_1, c_2, $\mathbf{P}_0$) affect the shape. Miura [10] proves that this family of curves is self-affine, i.e., any 'tail' of a given curve can be affinely transformed to represent the whole, so we can discount any affine transformations introduced by the parameters. It is easy to see that $\mathbf{P}_0$ and c_2 control the starting position and tangent, respectively. Altering c_1 selects the starting s value (since $c_0 s + (c_1 + \Delta c_1) = c_0(s + \Delta c_1/c_0) + c_1 = c_0\hat{s} + c_1$), while c_0 and c_1 are jointly responsible for the scaling. Similarly to the 'standard form' of [14], let us set these parameters such that

$$\mathbf{C}(0) = \mathbf{0}, \qquad \theta(0) = 0, \qquad \kappa(0) = 1, \qquad \kappa'(0) = 1. \tag{11}$$

It is evident then that the α parameter—the slope of the LCH—is what defines the shape of the whole curve. Figure 2 shows some examples with different α values.

The special case of $\alpha = 0$ neatly fits between the curves of small positive and negative α values on the left figure, but there is no place for it on the right. On the other hand, the $\alpha \to \pm\infty$ limit curve appears as the unit circle on the right, while the same curve acts only as the limit of $\alpha \to -\infty$ on the left.

3 GENERALIZED CATENARIES

We have seen that a natural requirement for aesthetic curves is that the curvature—or, as in log-aesthetic curves, a *power* of the curvature—should be a simple and smooth function, such as a straight line. A straightforward generalization would also allow parabolas:

$$\kappa(s) = (c_0 s^2 + c_1 s + c_2)^{-1/\alpha}. \tag{12}$$

This is a special case of the σ-curve family [11]; an alternative generalization (where the LCH itself is quadratic) is introduced in [15].

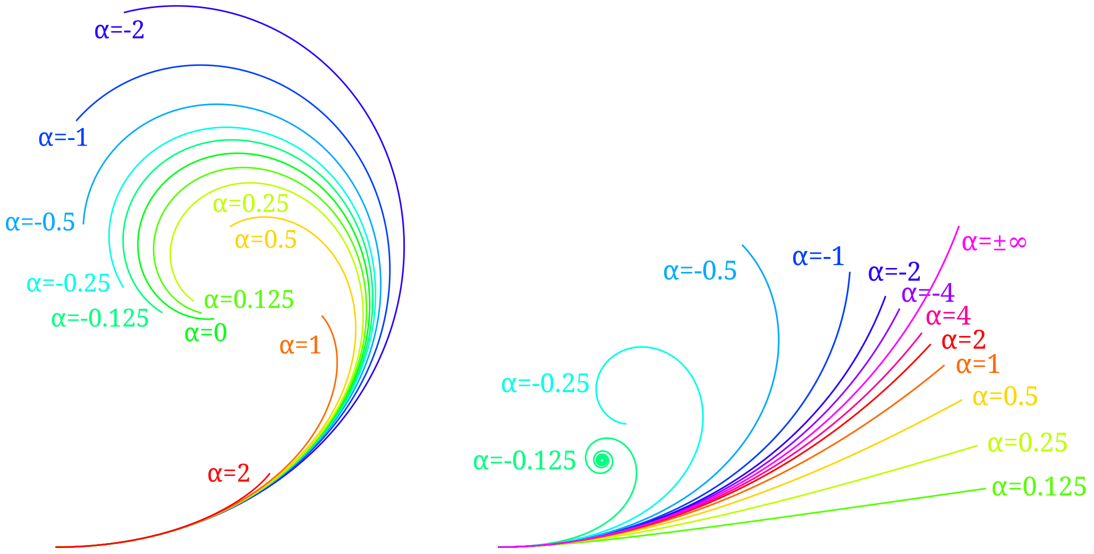


Figure 2: Log-aesthetic curves with various α values. Left: in standard form (and thus different c_0, c_1, c_2 parameters). Right: with fixed $c_0 = c_1 = 1$ and $\theta(0) = 0$.

When $c_0 = 0$, we get back Eq. (5), and also when $c_1 = 2\sqrt{c_0 c_2}$ it can be converted to the form of Eq. (5):

$$\kappa(s) = (\sqrt{c_0}s + \sqrt{c_2})^{-\frac{1}{\alpha/2}}. \tag{13}$$

But other cases are significantly different, and this broader class of curves includes *catenaries*. The name comes from Latin *catena* (chain), as it describes the shape a hanging chain takes when supported at its endpoints. It has long been regarded as an aesthetic curve, and it is used in architecture for arches and domes. The definition is usually given as the graph of the function

$$y = a\cosh(x/a), \tag{14}$$

where a is a shape parameter. Its Cesàro equation is

$$\kappa(s) = \frac{a}{s^2 + a^2}, \tag{15}$$

which means $c_0 = \frac{1}{a}$, $c_1 = 0$, $c_2 = a$ and $\alpha = 1$ in Eq. (12).

The tangent angle for $\alpha = 1$ is

$$\theta(s) = D \cdot \arctan((c_0 s + \frac{1}{2}c_1)D) + c_3, \qquad \text{with } D = (c_0 c_2 - \frac{1}{4}c_1^2)^{-\frac{1}{2}}, \tag{16}$$

which simplifies to $\theta(s) = \arctan(s/a) + c_3$ in the case of catenaries. For general α values the equation for $\theta(s)$ gets fairly complex or even intractable. The case of $\alpha = -1$, however, presents no difficulties. An interesting subfamily is one where—similarly to catenaries—c_1 is set to 0. As before, set $\kappa(0) = 1$ and $\theta(0) = 0$, so the equation becomes

$$\kappa(s) = c \cdot s^2 + 1, \qquad \theta(s) = \frac{1}{3}c \cdot s^3 + s. \tag{17}$$

For positive c values, the curves resemble hyperbolic spirals, although their pitch angle is not quite right. (We will return to the topic of hyperbolic spirals and pitch angles in the next section.) For negative c values, they start off somewhat similarly to elastica, but while the curvature of the spirals steadily increases, that of the elastica is periodic. With this in mind, we should turn next, in our quest for aesthetic curves, to periodic curvature functions.

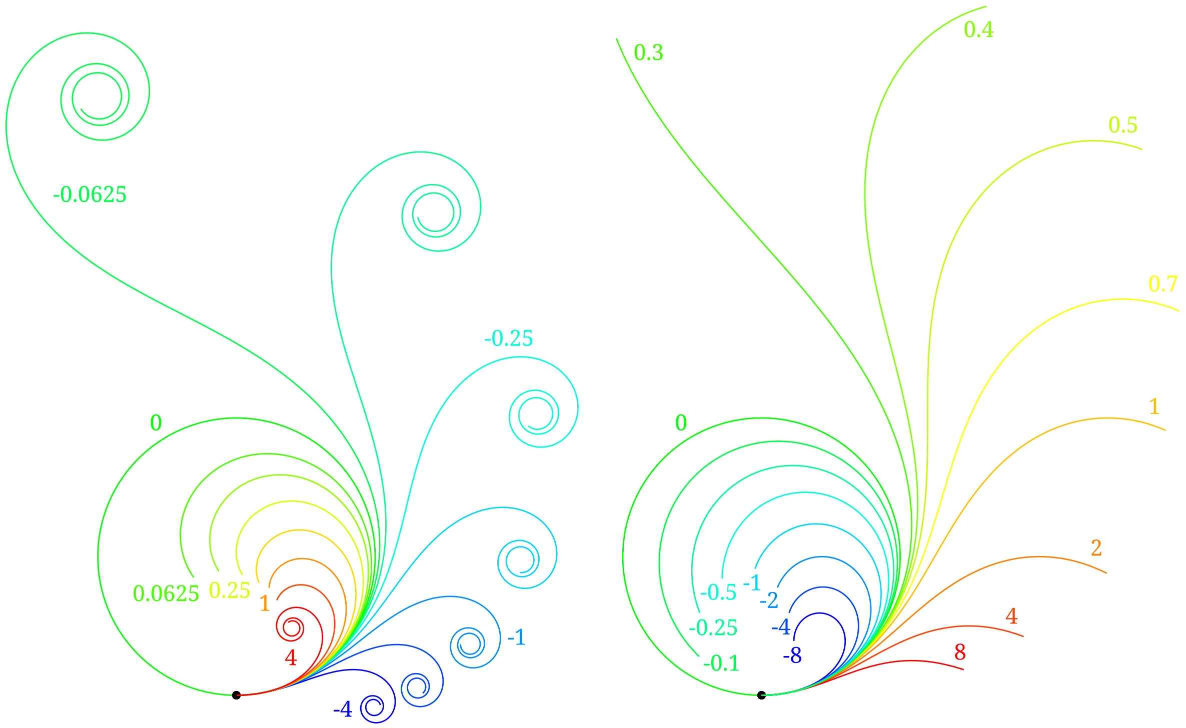


Figure 3: Left: Curves with curvature $\kappa(s) = c \cdot s^2 + 1$. Values for c set as $\pm 2^k$, $k \in \{-4, -3, -2, -1, 0, 1, 2\}$, showing also the $c = 0$ circle. Right: *Elastica* – solutions to the differential equation $\theta''(s) + \lambda \sin \theta(s) = 0$ for various λ values.

4 TRIG-AESTHETIC CURVES

Curves whose curvature is a periodic function of arc length were studied by Eduard Lehr in his dissertation [8]. Cosine is probably the simplest periodic function we can use as curvature. We take Eq. (5) and replace exponentiation with cosine; and since we want to be able to express curvatures of any magnitude, we also multiply it by a constant, arriving at

$$\kappa(s) = c_0 \cos(c_1 s + c_2), \qquad \theta(s) = \frac{c_0}{c_1} \sin(c_1 s + c_2) + c_3. \tag{18}$$

This turns out to be a curve used in geophysics (by the name *sine-generated curves* [5]) to model *river meandering*, the wiggling shape a river takes as a result of erosion, transportation and deposition [7]. This natural process is actually connected with elastica, since this is the most probable path particles take when their turning angles are defined by a normal distribution [13], but—as we will see—the curve defined above is a good approximation.

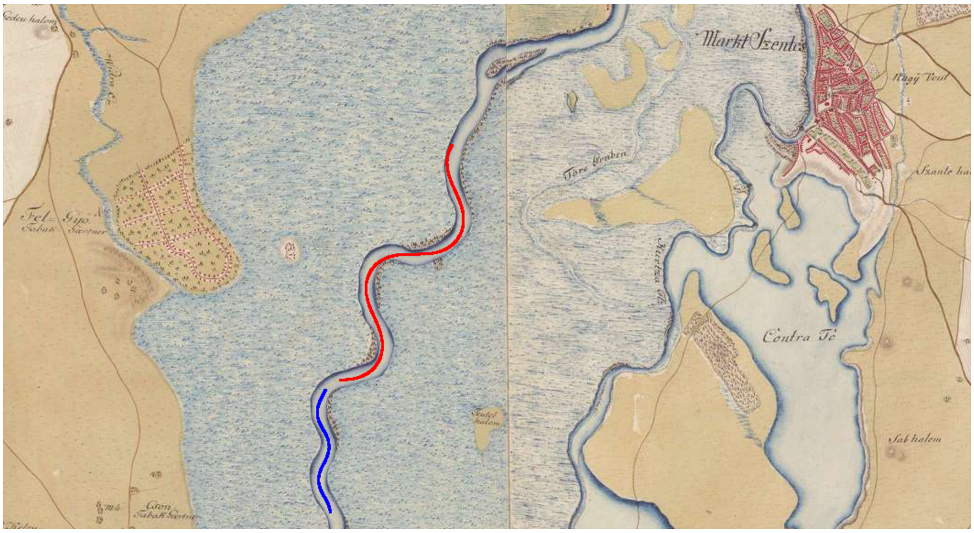


Figure 4: Meanders of the Tisza river, near Szentes (Hungary), before its regulation in the XIX. century (source: Arcanum Maps [1]). Two segments are approximated with the curve defined in Eq. (18), $c_0 = 1$, $c_1 \in \{1, 2\}$, $c_2 = 0$.

Following Langbein and Leopold [7], let us start with the elastica equation:

$$\theta''(s) + \lambda \sin\theta(s) = 0, \qquad \text{(multiply by } \theta'(s)\text{)} \tag{19}$$

$$\theta''(s)\theta'(s) + \lambda \sin\theta(s) \cdot \theta'(s) = 0, \qquad \text{(integrate)} \tag{20}$$

$$\frac{1}{2}\theta'(s)^2 - \lambda \cos\theta(s) = C, \tag{21}$$

which gives a family of curves based on C. When $|C| \leq |\lambda|$, we can define

$$\cos\omega = -\frac{C}{\lambda}, \tag{22}$$

and then

$$\theta'(s) = \sqrt{2(C + \lambda\cos\theta(s))} = \sqrt{2\lambda(\cos\theta(s) - \cos\omega)} = \sqrt{2\lambda(1-\cos\omega)\frac{\cos\theta(s) - \cos\omega}{1-\cos\omega}} \tag{23}$$

$$= \sqrt{2\lambda(1-\cos\omega)\left(1 - \frac{\cos\theta(s) - 1}{\cos\omega - 1}\right)}. \tag{24}$$

Now, since

$$\frac{\cos\alpha - 1}{\cos\beta - 1} = \frac{-\alpha^2/2 + O(\alpha^4)}{-\beta^2/2 + O(\beta^4)} \approx \left(\frac{\alpha}{\beta}\right)^2, \tag{25}$$

we have

$$\theta'(s) \approx \sqrt{2\lambda(1-\cos\omega)\left(1 - \left(\frac{\theta(s)}{\omega}\right)^2\right)}. \tag{26}$$

This approximative formula has an explicit solution:

$$\theta(s) = \omega \sin\left(\frac{\sqrt{2\lambda(1-\cos\omega)}}{\omega}s\right) = \frac{c_0}{c_1}\sin(c_1 s), \tag{27}$$

which is the same as Eq. (18) when $c_2 = c_3 = 0$.

The c_i constants have simple geometric meanings. Starting from the last one, c_3 controls the starting tangent, or in other words, rotates the curve; c_2 sets the starting parameter, so changing it shifts through the curve; and we can scale the curve by dividing both c_0 and c_1 by the scaling factor. The only parameter that really affects the *shape* is c_0, so in order to simplify our investigation, let us define *trig-aesthetic* curves as

$$\kappa(s) = \cos(s/c), \qquad \theta(s) = c\sin(s/c). \tag{28}$$

This c parameter also has a geometric meaning: it is the maximum angle the curve can deviate from the starting tangent. (This angle is $\arccos(1-\frac{1}{2\lambda})$ for the elastica, as can be derived from the equation of a pendulum with unit mass [12].)

The differential equation form of this curve family is

$$\theta''(s) + \theta(s)/c^2 = 0, \tag{29}$$

which differs only by a sign from that of the LA curves when $\alpha = 0$, $c_0 = 1/c$ and $c_2 = 0$:

$$\theta''_{\mathrm{LA}}(s) - \theta_{\mathrm{LA}}(s)/c^2 = 0. \tag{30}$$

If we also allow complex numbers for the c constant, using $c = -i$ results in

$$\kappa(s) = \cos(-s/i) = \cos(is) = \cosh(s), \qquad \theta(s) = \sinh(s). \tag{31}$$

The differential equation form now becomes the same as for Nielsen's spiral (Eq. (30) with $c = 1$), although initial values differ. This curve resembles the one in Eq. (17) with $c = \frac{1}{2}$, since $\cosh(s) \approx \frac{1}{2}s^2 + 1$ for small s values (easily seen from the Taylor expansion of $\cosh(s)$), and thus it is also similar to a hyperbolic spiral.

Spirals can be described by their *pitch angle*. At a given point of the spiral, draw a circle around the same center that touches that point. The pitch is the angle between the tangent of the spiral and the tangent of the circle. For example, logarithmic spirals have a constant pitch angle, while for Archimedean spirals it decreases as we get farther away from the center. In the case of hyperbolic spirals, the tangent of the pitch angle is proportional to the radius, i.e., the distance to the center.

Generally, for a curve with tangent angle $\theta(s)$ and point $C(s)$ in the complex plane, and P_0 set s.t. the spiral center is at the origin (i.e., $C(\infty) = 0$ for converging spirals), the pitch angle is

$$\beta = \arccos\left(\mathrm{Re}\left(\frac{C(s)}{|C(s)|}i \cdot \overline{\exp(i\theta(s))}\right)\right). \tag{32}$$

The central position for the curve in Eq. (31) can be computed as

$$K_0(1) + \frac{\pi}{2}(I_0(1) - \mathbf{L}_0(1))i \approx 0.421024 + 0.873084i, \tag{33}$$

where I_0 and K_0 are modified Bessel functions, and $\mathbf{L}_0$ is the modified Struve function. Numerical computation shows that $(\tan\beta)/r$ converges to 1, so the curve is indeed, in a sense, an approximation to the hyperbolic spiral.

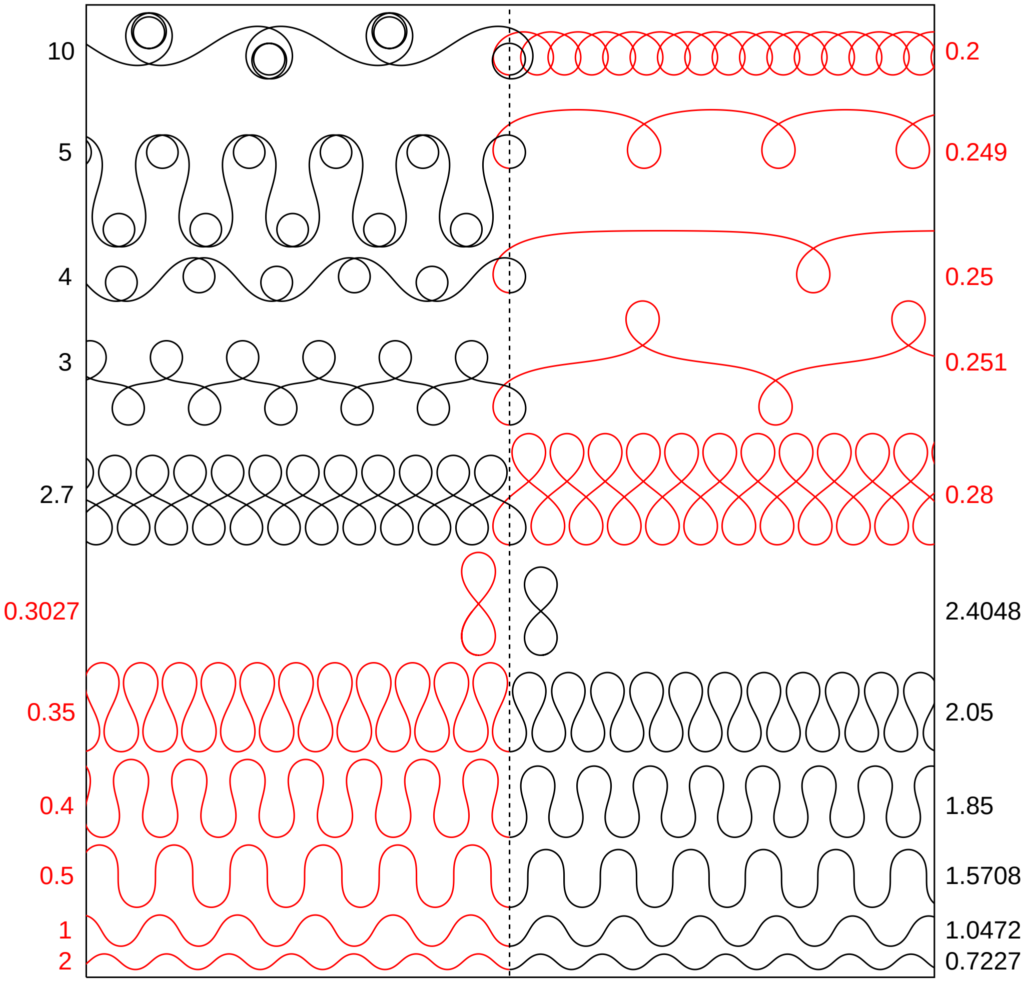


Figure 5: Elastica (red) vs. trig-aesthetic curves (black) for various (λ and c) parameter values. For $\lambda > 0.25$ the corresponding c values were selected such that the curves should exhibit similar behavior (but not necessarily the same maximum angle). For $\lambda \leq 0.25$ there are no correspondences; the parameters are chosen to showcase the different shapes. This figure was inspired by Fig. 11 in [9].

The similarity between the two can also be shown using the LCH slope, computed as

$$\alpha(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right) = 1 - \frac{\rho(s)\rho''(s)}{\rho'(s)^2}, \qquad \text{(see [3])} \tag{34}$$

which becomes

$$\alpha_{\text{trig\_aesthetic}}(s) = -1 - \cot^2(s/c) \tag{35}$$

for trig-aesthetic curves, and thus $-1 + \coth^2(s)$ when $c = -i$. This latter function quickly approaches 0, as does the slope of the hyperbolic spiral [16], see Figure 6.

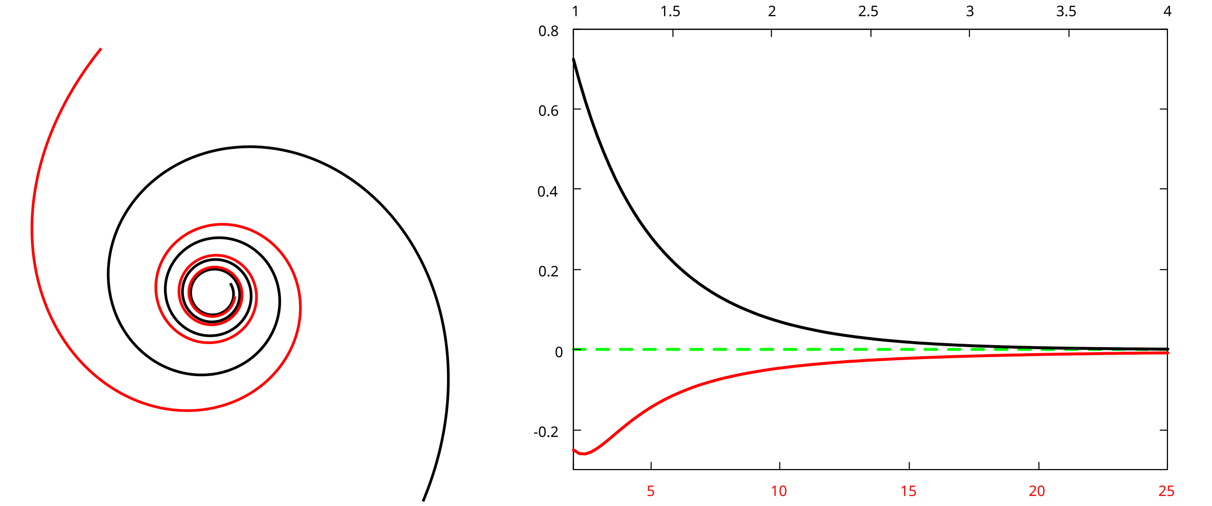


Figure 6: Left: Comparison of the hyperbolic spiral $r = 1/\varphi$ (red, $\varphi \in [2, 25]$) and a trig-aesthetic curve with $c = -i$ (black, $s \in [1, 4]$). Right: LCG slopes of the two curves.

5 HERMITE INTERPOLATION

Trig-aesthetic curves can express a wide variety of shapes, making them suitable for design. But a designer needs an intuitive tool for drawing instead of an abstract mathematical representation. It is essential, then, to be able to fit segments of these curves on user-specified end conditions, such as positions and tangent directions.

Since we can control translation, rotation and scaling by (respectively) $\mathbf{P}_0$, c_2 and c_1, we can assume that the starting point $\mathbf{Q}_0$ is at the origin and that the starting angle points in the direction of the x axis. The distance of the endpoint $\mathbf{Q}_1$ is also irrelevant, so the interpolation problem is defined by just two angles: ψ, the angle between $\overline{\mathbf{Q}_0\mathbf{Q}_1}$ and the x axis, and $\Delta\theta$, the change in tangent between the two endpoints (see Fig. 7).

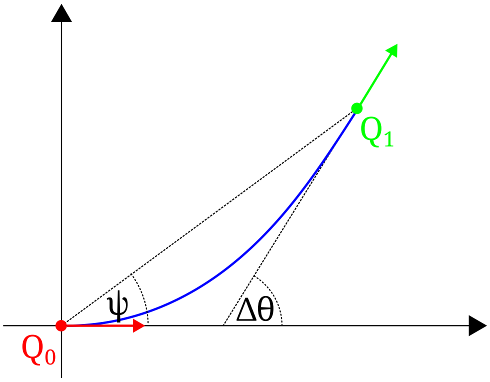


Figure 7: Simplified Hermite interpolation problem.

We are going to use the formula in Eq. (28). As the trig-aesthetic curve is not self-affine, it *does* matter which segment we take. In other words, the parameter interval $[s_0, s_1]$ introduces two additional degrees of freedom. Assuming that the shape parameter c has some fixed, user-defined value, we need to satisfy

$$\Delta\theta = \mathrm{mod}(\theta(s_1) - \theta(s_0) + \pi, 2\pi) - \pi \tag{36}$$
$$= \mathrm{mod}(c\sin(s_1/c) - c\sin(s_0/c) + \pi, 2\pi) - \pi, \tag{37}$$
$$\psi = \mathrm{mod}(\mathrm{atan2}(\mathrm{Im}(C(s_1) - C(s_0)), \mathrm{Re}(C(s_1) - C(s_0))) - \theta(s_0) + \pi, 2\pi) - \pi \tag{38}$$
$$= \mathrm{mod}\left(\mathrm{atan2}\left(\int_{s_0}^{s_1} \sin(c\sin(s/c))\,\mathrm{d}s, \int_{s_0}^{s_1} \cos(c\sin(s/c))\,\mathrm{d}s\right) - c\sin(s_0/c) + \pi, 2\pi\right) - \pi, \tag{39}$$

where $\mathrm{atan2}(y,x) = \mathrm{Im}\log(x+iy)$ is the two-argument arctangent, and $\mathrm{mod}(x,y) = x - y\lfloor x/y \rfloor$ is the floor-based modulo operation. Note that both $\Delta\theta$ and ψ are in the $[-\pi, \pi]$ range.

If we know s_0, the correct value for s_1 can be calculated as follows:

$$s_1 = s_0 + \min(\mathrm{mod}(\hat{s}_1 - s_0, 2\pi c), \mathrm{mod}(\pi c - \hat{s}_1 - s_0, 2\pi c)), \quad \text{where} \tag{40}$$
$$\hat{s}_1 = c\arcsin(\mathrm{mod}(\sin(s_0/c) + (\Delta\theta + \pi)/c, 2\pi/c) - \pi/c). \tag{41}$$

We will find s_0 by binary search, similarly to the algorithm of Yoshida & Saito [14]. The error function is the deviation between the left- and right-hand sides of Eqs. (38)–(39). The initial bracket for s_0 can be found by sampling the $[0, 2\pi c]$ interval; care should be taken because ψ values are 2π-periodic.

Multiple segments of the same curve can serve as solutions to a given interpolation problem. We can choose one by defining a *fairness* measure, such as

$$E_\kappa = \frac{1}{\mu^2}\int_{s_0}^{s_1} \kappa(s)^2\,\mathrm{d}s = \frac{1}{4\mu^2}\left[2s + c\sin(2s/c)\right]_{s_0}^{s_1}, \tag{42}$$

where

$$\mu = \frac{\|\mathbf{Q}_1 - \mathbf{Q}_0\|}{\|\mathbf{C}(s_1) - \mathbf{C}(s_0)\|} \tag{43}$$

is the scaling factor between the input data and the curve in standard form. A problem with this energy is that it does not penalize large loops. A better measure may be based on arc length (see Figs. 8a–8b):

$$E_s = \mu(s_1 - s_0). \tag{44}$$

Since the c shape parameter controls the maximum deviation of the tangent, the problem cannot be solved when $|\Delta\theta| > c$. But larger c values often result in loops, so a further search for finding a minimal c value may be appropriate, see Figures 8c–8d.

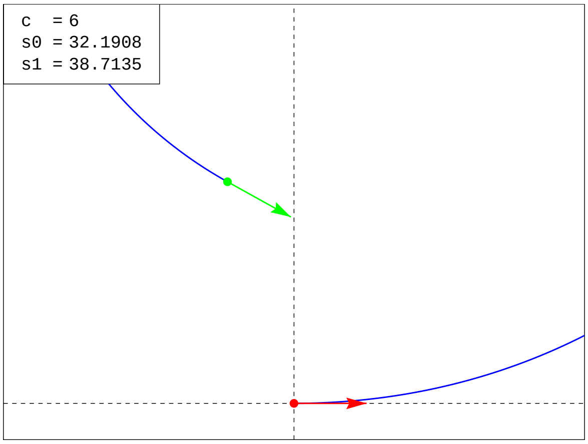
6 CONCLUSIONS

A generalization of log-aesthetic curves has been given, which includes catenaries. We have also defined a variant curve family, which we called *trig-aesthetic* curves. The 'aesthetic' attribute is appropriate, because:

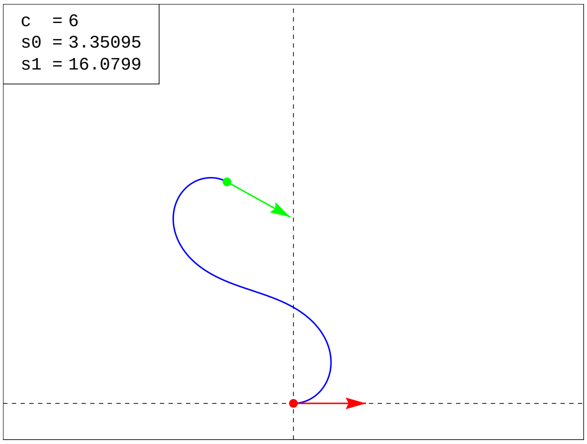
- It approximates (for a parameter range) the elastica family, which is an epitome of aesthetic curves.
- Its curvature is a simple, smooth function.
- It is very closely related to log-aesthetic curves, particularly to Nielsen's spiral.
- With complex parameters it approximates the hyperbolic spiral, another aesthetic curve.

It has also been shown that trig-aesthetic curves can be used for design using Hermite interpolation.

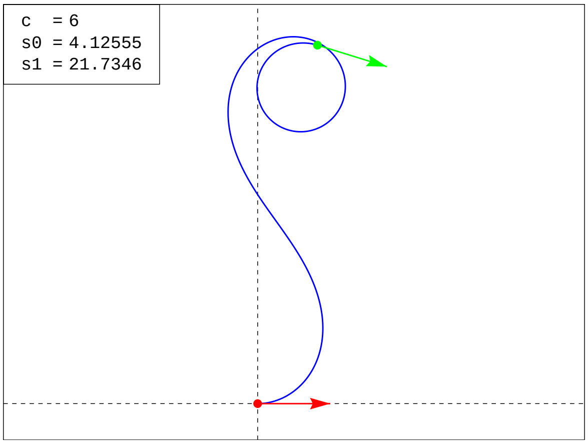
It would be interesting to further generalize these curves to include or approximate other spirals in the Archimedean family (i.e., those with polar equation $r = a + b\phi^{1/n}$, such as the arithmetic spiral, the lituus or Fermat's spiral).



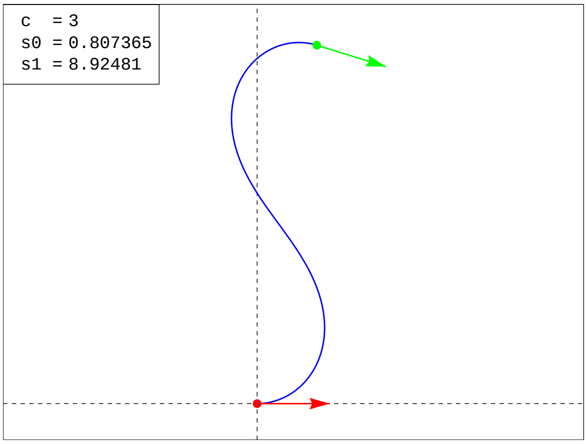
(a) Test case 1 – best fit by E_κ



(b) Test case 1 – best fit by E_s



(c) Test case 2 – best fit by E_s with $c = 6$



(d) Test case 2 – best fit by E_s with $c = 3$

Figure 8: Hermite fitting examples.

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