Polyhedral design with blended *n*-sided interpolants

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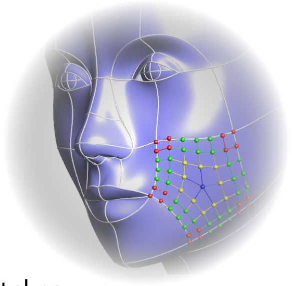
Outline



General idea

Regular meshes
- Quadratic Bézier patches
- Blending functions

Irregular vertices
- Quadratic Generalized Bézier (QGB) patches
- Parameterization
- Triangular patches

Results

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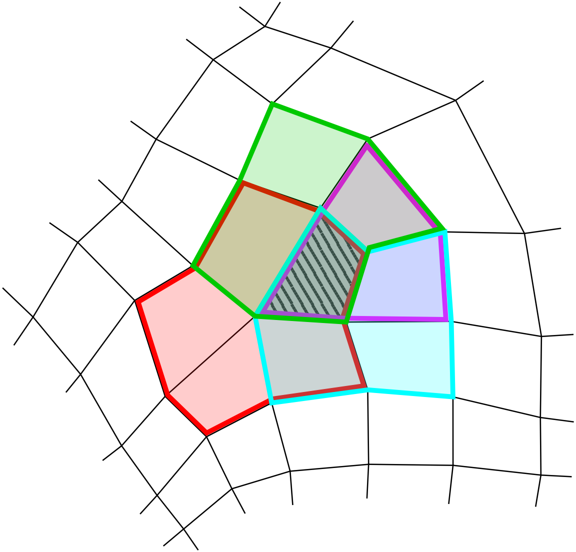
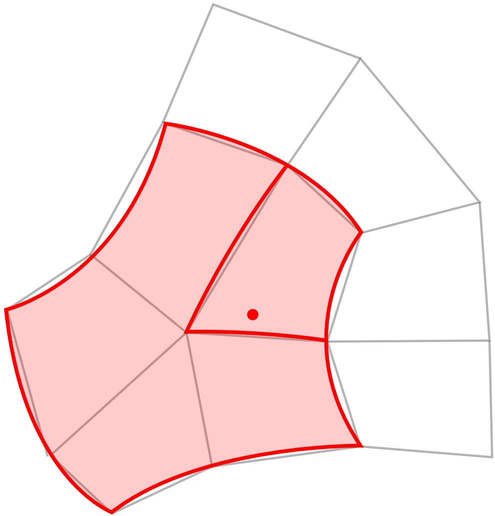
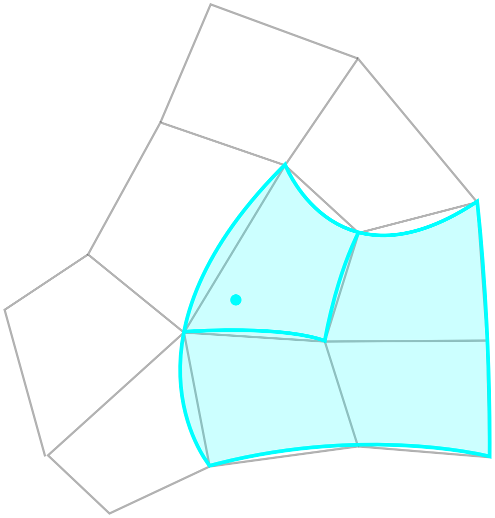
Construction plan



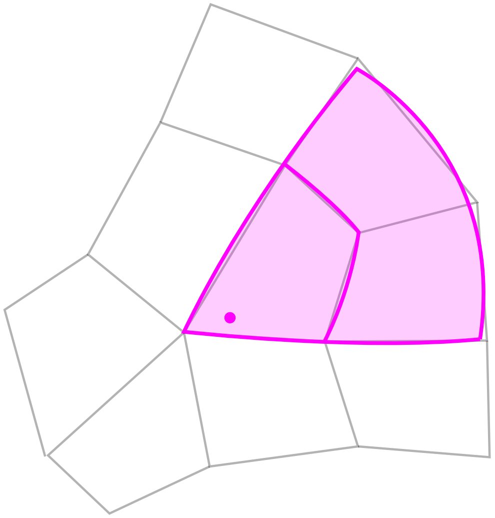
Interpolant #1



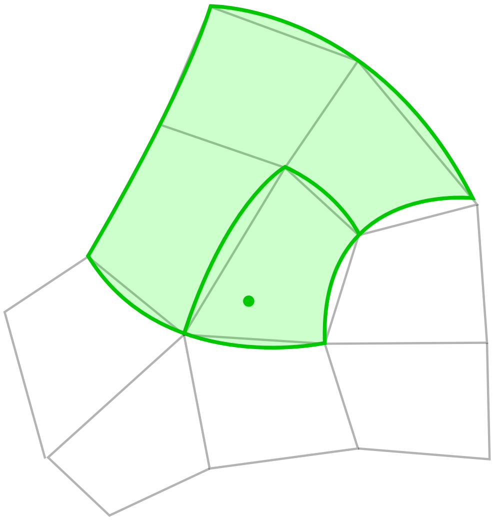
Interpolant #2



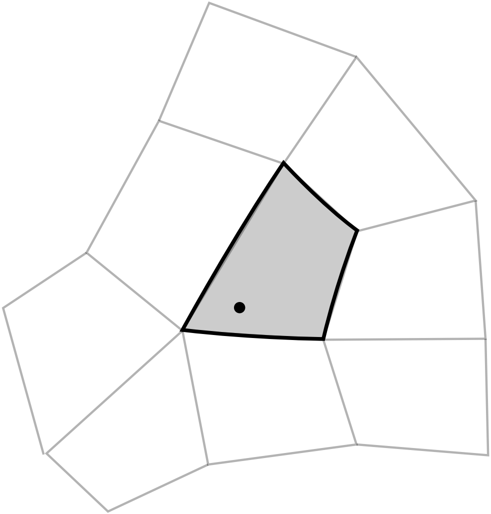
Interpolant #3



Interpolant #4



Blended patch



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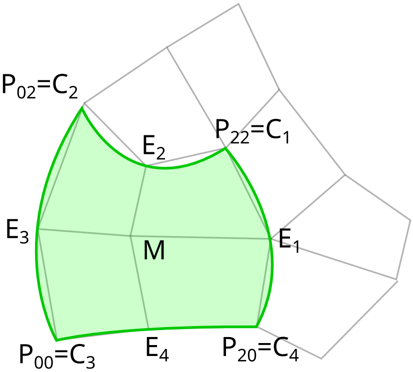
Quadratic Bézier patches



$$\mathbf{I}(u,v) = \sum_{i=0}^{2}\sum_{j=0}^{2} \mathbf{P}_{ij} B_i^2(u) B_j^2(v)$$

$$\mathbf{P}_{02} = \mathbf{C}_2 \quad \mathbf{P}_{12} = \hat{\mathbf{E}}_2 \quad \mathbf{P}_{22} = \mathbf{C}_1$$

$$\mathbf{P}_{01} = \hat{\mathbf{E}}_3 \qquad\qquad \mathbf{P}_{21} = \hat{\mathbf{E}}_1$$

$$\mathbf{P}_{00} = \mathbf{C}_3 \quad \mathbf{P}_{10} = \hat{\mathbf{E}}_4 \quad \mathbf{P}_{20} = \mathbf{C}_4$$

$$\hat{\mathbf{E}}_i = 2\mathbf{E}_i - \frac{1}{2}(\mathbf{C}_{i-1} + \mathbf{C}_i)$$

$$\mathbf{P}_{11} = \frac{1}{4}(16\mathbf{M} - \sum_{i=1}^{4}(\mathbf{C}_i + 2\hat{\mathbf{E}}_i))$$

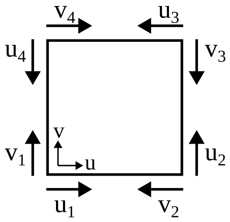
$$[0,1]^2 \to [0.5,1]^2 \quad \Rightarrow \quad \mathbf{S}(0,0) = \mathbf{M}, \quad \mathbf{S}(1,1) = \mathbf{C}_1, \text{etc.}$$

Patch equation

$$\mathbf{S}(u,v) = \sum_{i=1}^{4} \mathbf{I}_i\left(\frac{u_i+1}{2}, \frac{v_i+1}{2}\right)\Phi(u_i, v_i)$$

Local parameterization:

$$\begin{aligned} u_1 &= u & v_1 &= v \\ u_2 &= v & v_2 &= 1-u \\ u_3 &= 1-u & v_3 &= 1-v \\ u_4 &= 1-v & v_4 &= u \end{aligned}$$



Blends:

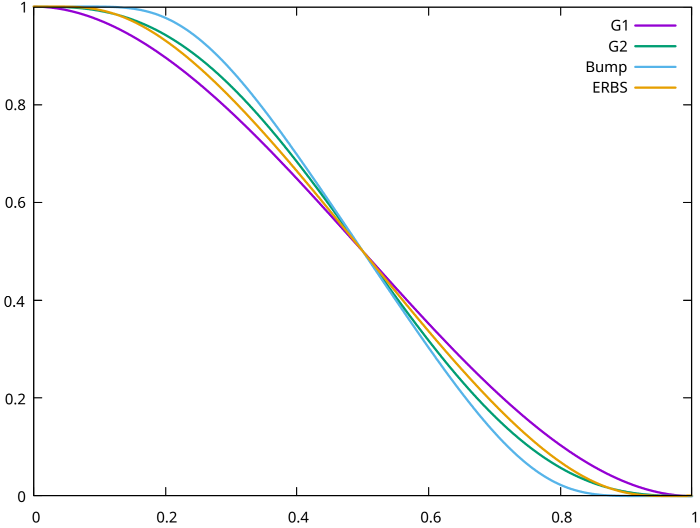
$$\Phi(u,v) = \Psi(u)\cdot\Psi(v)$$

$$\Psi(0) = 1 \quad \Psi(1) = 0 \quad \Psi^{(k)}(0) = \Psi^{(k)}(1) = 0$$

$$(\text{for some } k > 0)$$

Blending functions

G^k Hermite blend: $\Psi(t) = \sum_{i=0}^{k} B_i^{2k+1}(t)$

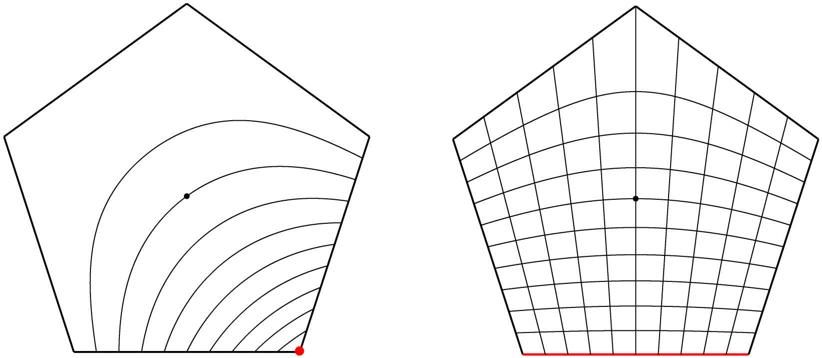


Outline

Quadratic Generalized Bézier patches – Parameterization

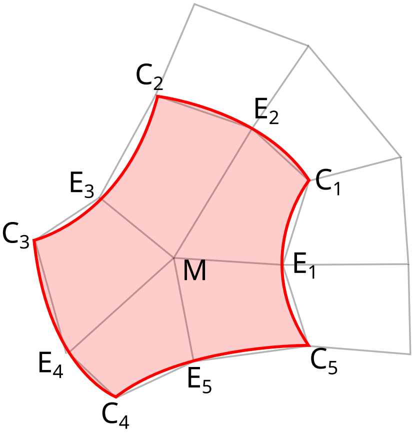
- Regular domain: $\{(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n})\}$, $k = 0 \ldots n-1$
- Wachspress coordinate-based local parameters:

$$s_i(u,v) = \lambda_i/(\lambda_{i-1} + \lambda_i), \quad d_i(u,v) = 1 - \lambda_{i-1} - \lambda_i$$



Quadratic Generalized Bézier patches – Equation

$$\mathbf{I}(u,v) = \sum_{i=1}^{n} \left(\mathbf{C}_{i-1}\frac{1}{2}B_0^2(s_i) + \hat{\mathbf{E}}_i B_1^2(s_i) + \mathbf{C}_i \frac{1}{2} B_2^2(s_i) \right) B_0^2(d_i) + \mathbf{P}_0 B_0(u,v)$$

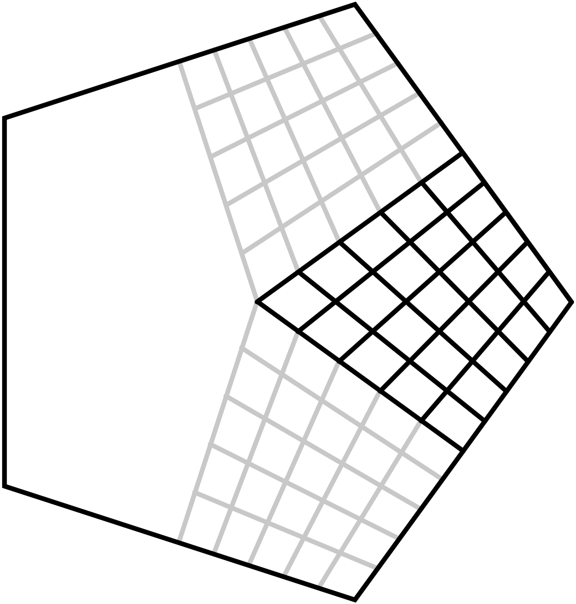


$$\mathbf{P}_0 = \frac{n^2\mathbf{M} - \sum_{i=1}^{n}\left(\mathbf{C}_i + 2\hat{\mathbf{E}}_i\right)}{n(n-3)}$$

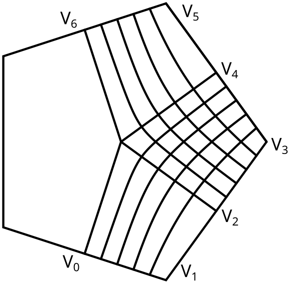
$$B_0 = 1 - \sum_{i=1}^{n}\left(\tfrac{1}{2}B_0^2 + B_1^2 + \tfrac{1}{2}B_2^2\right)B_0^2$$

- $n = 4 \Rightarrow$ Bézier patch
- $n = 3 \Rightarrow$ Bézier triangle

Parameter mapping – Problem



Parameter mapping – Solution



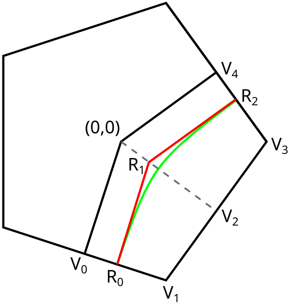
- Rational Bézier curves:

$$\mathbf{r}(t) = \frac{\sum_{i=0}^{2} \mathbf{R}_i w_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}$$

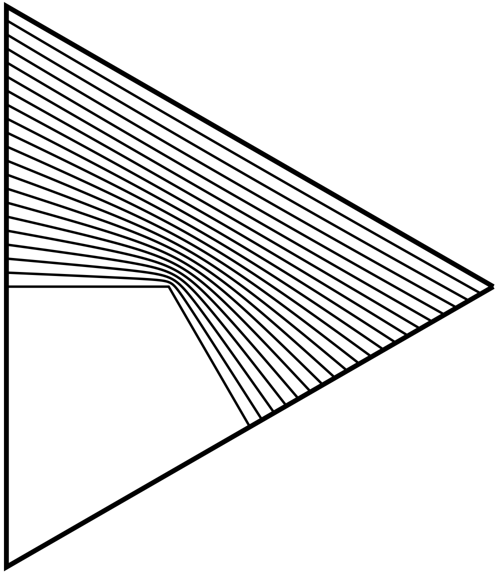
- Central weight: $1/u$
- Control points:

$$\mathbf{R}_0 = \mathbf{V}_0(1-u) + \mathbf{V}_1 u$$
$$\mathbf{R}_1 = \mathbf{V}_2 u \cdot e^{u(u-1)}$$
$$\mathbf{R}_2 = \mathbf{V}_4(1-u) + \mathbf{V}_3 u$$

- Intersection with golden section search



Parameter mapping – Concave case



Triangular patches

- Triangular QGB patch $\equiv$ quadratic Bézier triangle
- $\rightarrow$ no central control $\Rightarrow$ elevate to cubic:

$$\mathbf{I}(u,v) = \sum_{i+j+k=3} \mathbf{P}_{ijk} \frac{6}{i!j!k!} \lambda_1^i \lambda_2^j \lambda_3^k$$

where

$$\mathbf{P}_{300} = \mathbf{C}_1, \quad \mathbf{P}_{030} = \mathbf{C}_2, \quad \mathbf{P}_{003} = \mathbf{C}_3,$$
$$\mathbf{P}_{210} = \frac{1}{3}(\mathbf{C}_1 + 2\hat{\mathbf{E}}_2), \quad \mathbf{P}_{120} = \frac{1}{3}(\mathbf{C}_2 + 2\hat{\mathbf{E}}_2),$$
$$\mathbf{P}_{021} = \frac{1}{3}(\mathbf{C}_2 + 2\hat{\mathbf{E}}_3), \quad \mathbf{P}_{012} = \frac{1}{3}(\mathbf{C}_3 + 2\hat{\mathbf{E}}_3),$$
$$\mathbf{P}_{102} = \frac{1}{3}(\mathbf{C}_3 + 2\hat{\mathbf{E}}_1), \quad \mathbf{P}_{201} = \frac{1}{3}(\mathbf{C}_1 + 2\hat{\mathbf{E}}_1),$$
$$\mathbf{P}_{111} = \frac{1}{6}(27\mathbf{M} - \sum_{\max(i,j,k)=3} \mathbf{P}_{ijk} - 3 \sum_{\max(i,j,k)=2} \mathbf{P}_{ijk})$$

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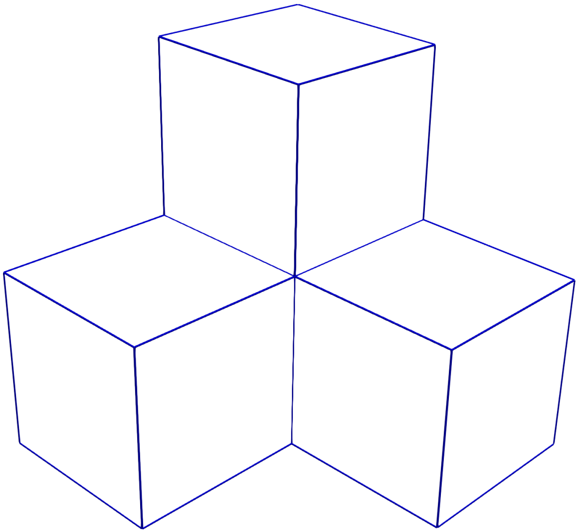
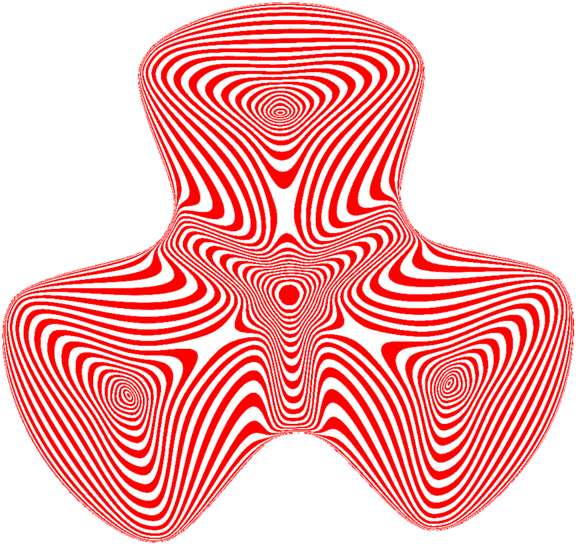
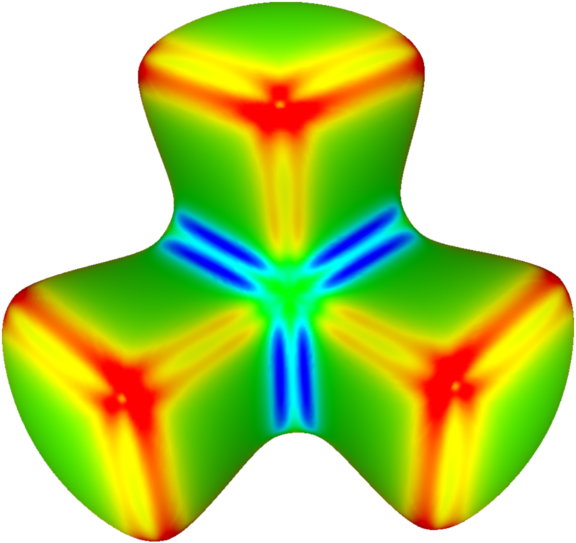
Trebol model – Cage



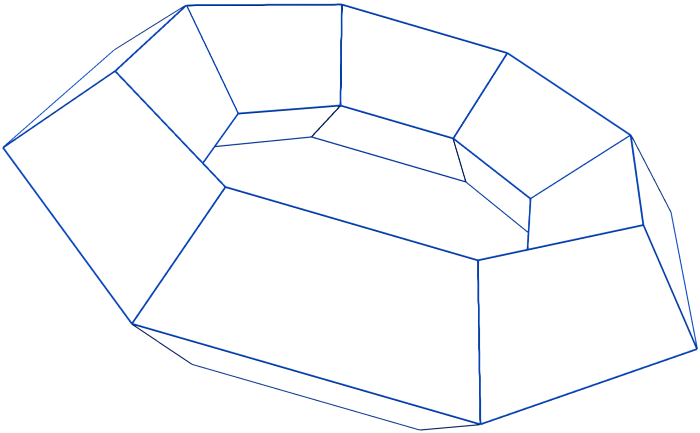
Trebol model – Isophotes



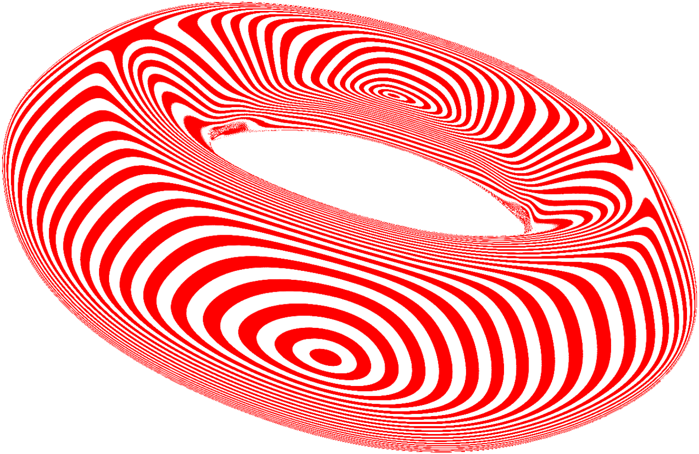
Trebol model – Mean curvature



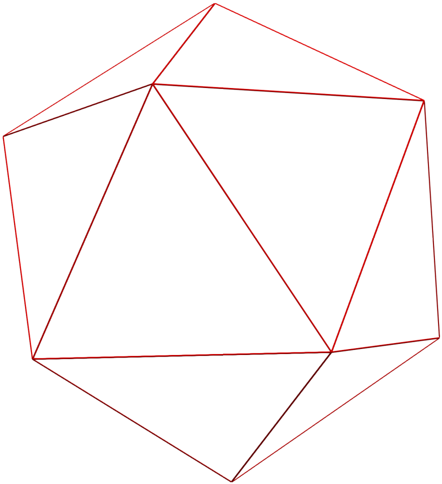
Torus model – Cage



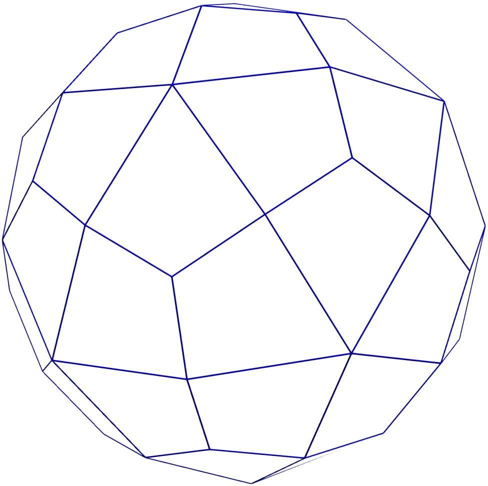
Torus model – Isophotes



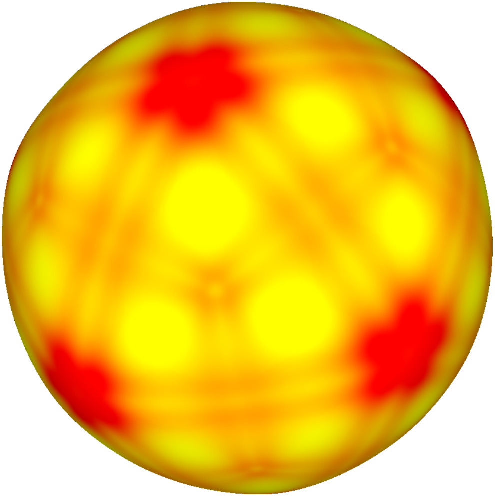
Icosahedron model – Cage



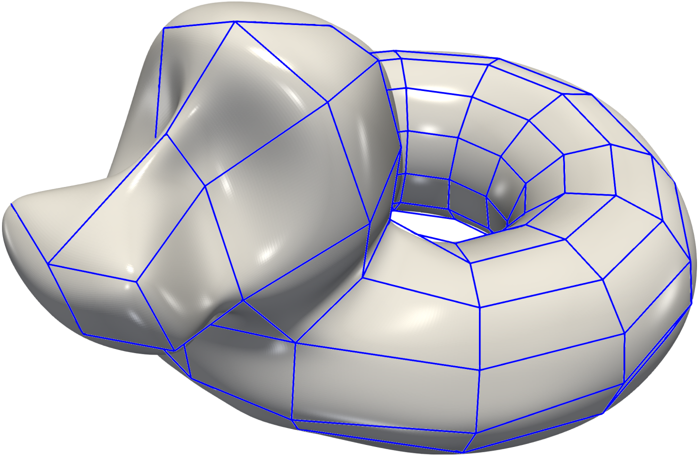
Icosahedron model – Cage after 1 Catmull–Clark step



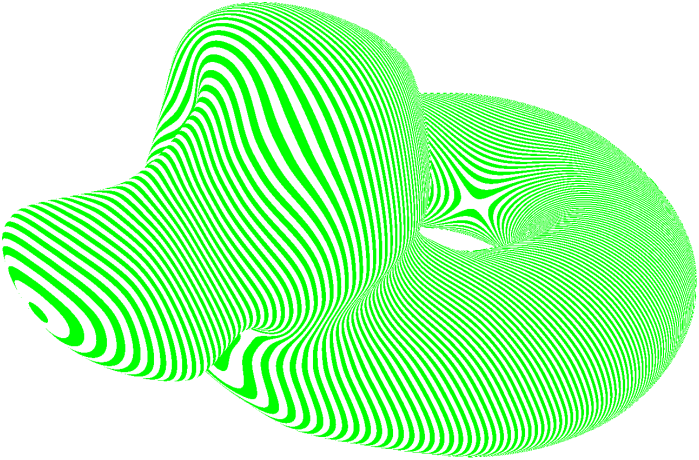
Icosahedron model – Mean curvature



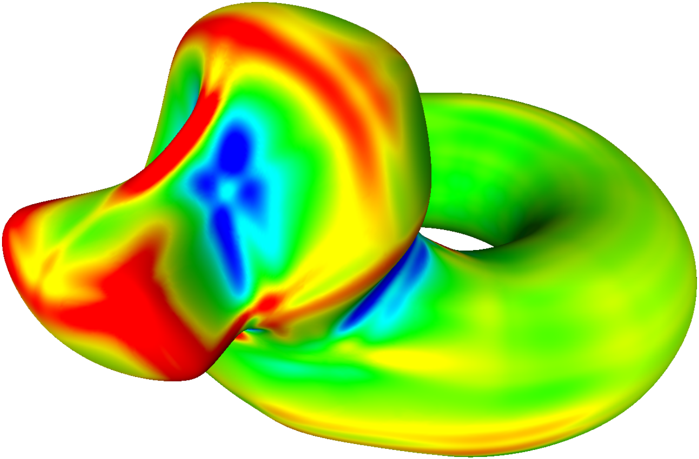
Duck model – Cage & surface



Duck model – Slicing

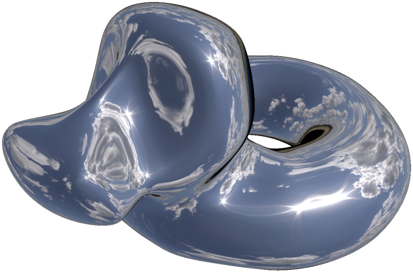


Duck model – Mean curvature



Conclusions & future work

- Parametric surface
- Interpolates mesh vertices of arbitrary topology
- Meshes with boundary?
- Continuous boundary constraints?
- Shape parameters?
- Normal vector interpolation?



https://3dgeo.iit.bme.hu/