Aesthetic curve families in computer-aided design

Péter Salvi

Budapest University of Technology and Economics

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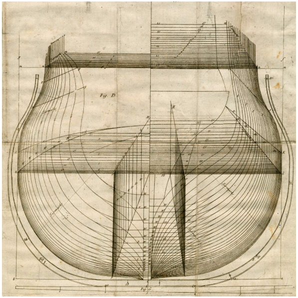
Outline

Triskelion motif in Newgrange, Ireland from the Neolithic Period (c. 3200 BC)

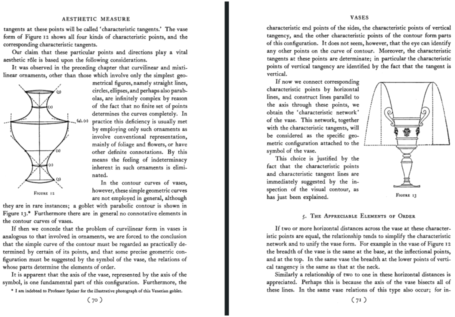


W. Sutherland: *The Shipbuilders Assistant*. London, 1755.

Aesthetic measure

Birkhoff on the curvature of vase contours:

- Curvature should vary continuously
- Curvature should not oscillate more than once
- The maximum rate of change of curvature should be minimal



AESTHETIC MEASURE

tangents at these points will be called 'characteristic tangents.' The vase form of Figure 12 shows all four kinds of characteristic points, and the corresponding characteristic tangents.

Our claim that these particular points and directions play a vital aesthetic rôle is based upon the following considerations.

It was observed in the preceding chapter that curvilinear and mixtilinear ornaments, other than those which involve only the simplest geometrical figures, namely straight lines, circles, ellipses, and perhaps also parabolas, are infinitely complex by reason of the fact that no finite set of points determines the curves completely. In practice this deficiency is usually met by employing only such ornaments as involve conventional representation, mainly of foliage and flowers, or have other definite connotations. By this means the feeling of indeterminacy inherent in such ornaments is eliminated.

FIGURE 12

In the contour curves of vases, however, these simple geometric curves are not employed in general, although they are in rare instances; a goblet with parabolic contour is shown in Figure 13.* Furthermore there are in general no connotative elements in the contour curves of vases.

If then we concede that the problem of curvilinear form in vases is analogous to that involved in ornaments, we are forced to the conclusion that the simple curve of the contour must be regarded as practically determined by certain of its points, and that some precise geometric configuration must be suggested by the symbol of the vase, the relations of whose parts determine the elements of order.

It is apparent that the axis of the vase, represented by the axis of the symbol, is one fundamental part of this configuration. Furthermore, the

* I am indebted to Professor Speiser for the illustrative photograph of this Venetian goblet.

(70)

VASES

characteristic end points of the sides, the characteristic points of vertical tangency, and the other characteristic points of the contour form parts of this configuration. It does not seem, however, that the eye can identify any other points on the curve of contour. Moreover, the characteristic tangents at these points are determinate; in particular the characteristic points of vertical tangency are identified by the fact that the tangent is vertical.

If now we connect corresponding characteristic points by horizontal lines, and construct lines parallel to the axis through these points, we obtain the 'characteristic network' of the vase. This network, together with the characteristic tangents, will be considered as the specific geometric configuration attached to the symbol of the vase.

This choice is justified by the fact that the characteristic points and characteristic tangent lines are immediately suggested by the inspection of the visual contour, as has just been explained.

FIGURE 13

5. THE APPRECIABLE ELEMENTS OF ORDER

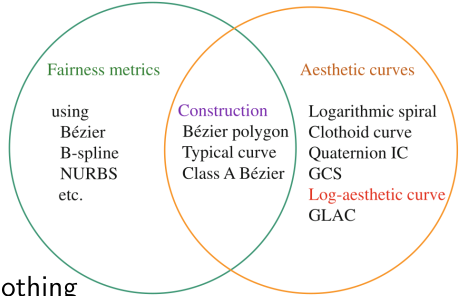
If two or more horizontal distances across the vase at these characteristic points are equal, the relationship tends to simplify the characteristic network and to unify the vase form. For example in the vase of Figure 12 the breadth of the vase is the same at the base, at the inflectional points, and at the top. In the same vase the breadth at the lower points of vertical tangency is the same as that at the neck.

Similarly a relationship of two to one in these horizontal distances is appreciated. Perhaps this is because the axis of the vase bisects all of these lines. In the same vase relations of this type also occur; for in-

(71)

G. D. Birkhoff: *Aesthetic Measure*. Harvard University Press, Cambridge, 1933.

Fair curves



Farin's Definition

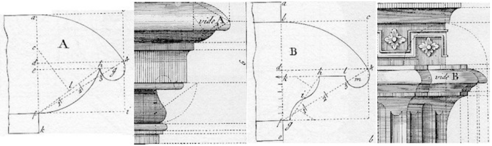
A curve is fair if its curvature plot is continuous and consists of only a few monotone pieces.

- Generic curves require smoothing
  - by post-processing
  - by variational fitting techniques
- Curve representations with intrinsic smoothness?
  - Curves with monotone curvature plots
  - Limit our scope to 2D curves
- Cesàro equation
  - Curvature as a function of arc length: $\kappa(s)$

G. Farin, G. Rein, N. Sapidis, A. J. Worsey: *Fairing cubic B-spline curves*. CAGD 4(1–2):91–103, 1987.
K. T. Miura, R. U. Gobithaasan: *Aesthetic design with log-aesthetic curves and surfaces*. In: Y. Dobashi, H. Ochiai (eds.): *Mathematical Progress in Expressive Image Synthesis III*, Mathematics for Industry 24, pp. 107–119, 2016.

Circle

- Loved since the beginning of time
- Most basic curve
- Cesàro equation: $\kappa(s) = c$ (const)
- Prevalent in CAD (and everywhere)
- Its use is limited in itself
  - Combination of circular arcs and straight line segments
  - Only C^0 or G^1 continuity



B. & T. Langley: *Gothic Architecture*. London, 1742.

Neolithic cult symbol, Spain

Parabola

- Menaechmus (4th century BC)
- Not always monotone curvature ★
- CAD – quadratic Bézier curve
  - TrueType fonts
  - SVG (`Q` and `T` commands)
  - κ-curves ★
- Nice physical properties
  - Used in bridges, arches
  - Also in antennas, reflectors

Apollonius' *Conics* (in Arabic, IX. c.)

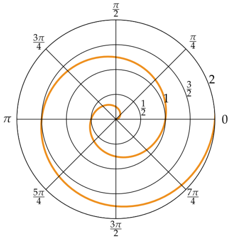
The Golden Gate bridge

A parabola antenna

Z. Yan, S. Schiller, G. Wilensky, N. Carr, S. Schaefer: *κ-curves: Interpolation at local maximum curvature*. ACM TOG 36(4):1–7, 2017.

Archimedes' (arithmetic) spiral

- Archimedes (3rd century BC)
  - Used for squaring the circle
- A line rotates with const. ω, and a point slides on it with const. v
- Polar equation: $r = a + b\phi$

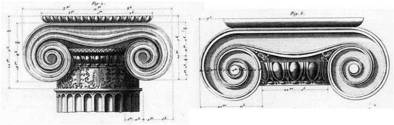


Source: Wikipedia (Archimedean spiral)

Great Mosque of Samarra, Iraq

Generalized Archimedean spiral

- Polar equation: $r = a + b\phi^{1/c}$
- $c = -2 \Rightarrow$ lituus (Cotes, XVIII. c.)
  - Augur's curved staff
  - Frequently used for volutes
- $c = -1 \Rightarrow$ hyperbolic spiral
- $c = 2 \Rightarrow$ Fermat's spiral

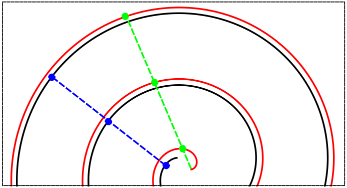


J. D. LeRoy: *Les ruines plus beaux des monuments de la Grèce*. Paris, 1758.

Crosier of Archbishop Heinrich of Finstingen

Circle involute

- Huygens (17th century)
  - Used for pendulum clocks
- Traced by the end of a rope coiled on a circular object ★
- Similar to Archimedes' spiral
  - But with constant normal spacing
- Cesàro equation: $\kappa(s) = c/\sqrt{s}$
- Used for cog profiles (since Euler) and scroll compressors (pumps)



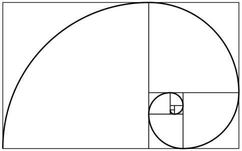
Archimedes' spiral & Circle involute

Hawaiian fern

Coiled millipede

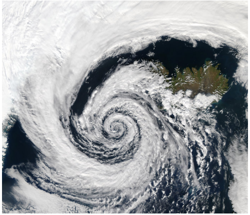
Logarithmic spiral

- Descartes & Bernoulli (XVII. c.)
  - "spira mirabilis"
- Polar equation: $r = ae^{b\phi}$
- The golden spiral is also logarithmic ($b = \frac{\ln \varphi}{\pi/2}$)
- Cesàro equation: $\kappa(s) = c/s$
- Very natural, self-similar pattern
  - Shells, sunflowers, cyclones etc.



Fibonacci spiral, approximating the golden spiral (Wikipedia)

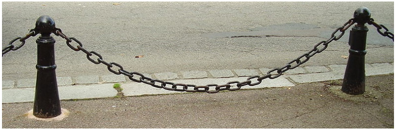
Lower part of Bernoulli's gravestone
(but the spiral is Archimedean)



Cyclone over Iceland (NASA)

Catenary curves

- Hooke; Leibniz, Huygens & Bernoulli (17th century)
  - Curve of a hanging chain or cable
- Equation: $y = a\cosh(x/a)$
- Cesàro equation: $\kappa(s) = a/(s^2 + a^2)$
- Used in architecture
  - Design of bridges / arches



A hanging chain showing a catenary curve

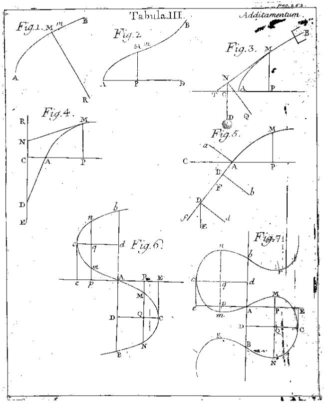
Gaudi's design of a church
at Santa Coloma de Cervello

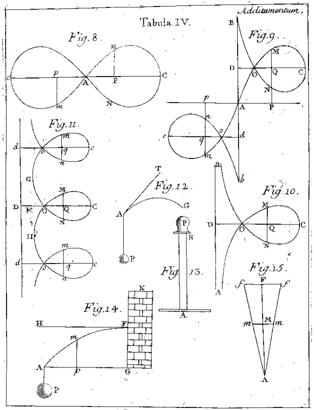
Spline energies

$$\kappa''(s) = 0 \qquad \text{(wooden)} \qquad \Rightarrow \text{clothoid}$$

$$\int \kappa(s)^2 \, \mathrm{d}s \to \min \qquad \text{(mechanical)} \qquad \Rightarrow \text{elastica}$$


Spline weights by Edson International



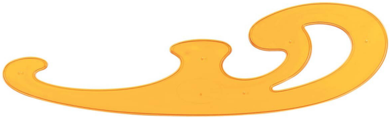


L. Euler: *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*. Lausanne & Geneve, 1744.

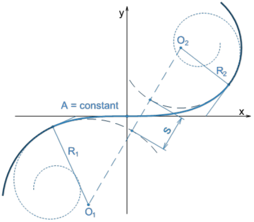
J. Hoschek, D. Lasser: *Fundamentals of Computer Aided Geometric Design*. A. K. Peters, Wellesley, 1996.

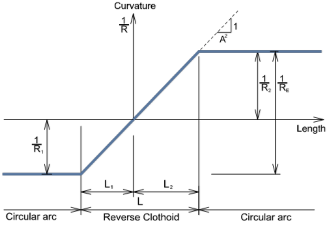
Clothoid (Euler/Cornu spiral)

- Euler (XVIII. c.) & Cornu (XIX. c.)
- G^2 transition between circular arcs and straight lines
- Cesàro equation: $\kappa(s) = c \cdot s$
- French curves have clothoid edges
- Used in urban planning
  - Railroad / highway design
  - Linear centripetal acceleration



A french curve





Source: PWayBlog

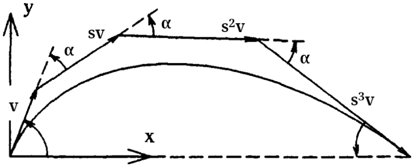
Outline

Typical Bézier Curves

- Bézier curves are *typical*, if each "leg" of the control polygon is obtained by the same rotation and scale of the previous one:

$$\Delta \mathbf{P}_{i+1} = s \cdot R \Delta \mathbf{P}_i \quad [\Delta \mathbf{P}_j = \mathbf{P}_{j+1} - \mathbf{P}_j]$$

  where s is the scale factor, R is a rotation matrix by α



- Class A Bézier curves are more general:

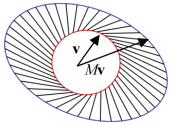
$$\Delta \mathbf{P}_i = M^i \mathbf{v}$$

  where M is a 2×2 matrix and $\mathbf{v}$ is a unit vector
- These curves can be extended to 3D, as well

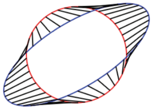
Y. Mineur, T. Lichah, J. M. Castelain, H. Giaume:
A shape controled fitting method for Bézier curves. CAGD 15(9):879–891, 1998.
G. Farin: *Class A Bézier curves.* CAGD 23(7):573–581, 2006.

Properties

- Goal: continuous & monotone curvature
- Typical curves need constraints on s and α
  - $\cos\alpha > 1/s$ (if $s > 1$) or $\cos\alpha > s$ (if $s \leq 1$)
- Class A Bézier curves need constraints on M
- Originally: the segments $\mathbf{v} - M\mathbf{v}$ do not intersect the unit circle for any unit vector v
- Corrected:
  - $M = SD^iS^{-1}$, where S is orthogonal, D is diagonal (assuming a symmetric M)
  - $d_{11} \geq 1$, $d_{22} \geq 1$,
    $2d_{11} \geq d_{22} + 1$,
    $2d_{22} \geq d_{11} + 1$
- Similar constraints for 3D curves



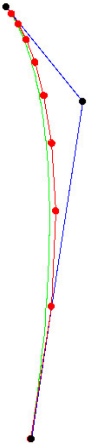
Matrix satisfying the constraints.


Matrix not satisfying the constraints.

J. Cao, G. Wang: *A note on Class A Bézier curves*. CAGD 25(7):523–528, 2008.

Interpolation

- We need: $\mathbf{P}_0$, $\mathbf{v}$ and the end tangent $\mathbf{v}_n$
  - Rotation angle $\alpha = \angle(\mathbf{v}, \mathbf{v}_n)/n$
  - Scale factor $s = (\|\mathbf{v}_n\|/\|\mathbf{v}\|)^{1/n}$
  - Condition: $\cos\alpha > 1/s$
    $\Rightarrow$ true if n is large enough
- Problems:
  - Cannot set the end position
    $\Rightarrow$ not designer-friendly
  - For $\|\mathbf{v}_n\| \approx \|\mathbf{v}\|$ the degree n must be very high
- Better input: position and tangent at both ends
  - Using 3 control points $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$



N. Yoshida, T. Saito: *Interactive control of planar Class A Bézier curves using logarithmic curvature graphs.* CAD&A 5(1-4):121–130, 2008.

Three-point interpolation ★

- Needed: $\mathbf{P}_0$, α, $\mathbf{v}$, s (assume fixed n)
- $\mathbf{P}_0 = \mathbf{a}_0$
- $\alpha = \angle(\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_1)/n$
- $\mathbf{v} = b_0 \cdot \frac{\mathbf{a}_1 - \mathbf{a}_0}{\|\mathbf{a}_1 - \mathbf{a}_0\|} =: b_0 \cdot \mathbf{u}$
- b_0 is defined by the equation

$$\sum_{j=0}^{n-1} b_0 M^j \mathbf{u} = \mathbf{a}_2 - \mathbf{a}_0, \quad M = s \cdot R(\alpha)$$

- For $n = 3$ this is a quadratic equation,
  otherwise polynomial root finding algorithms are needed
  $\Rightarrow$ just approximates the endpoint
- For large n these curves converge to logarithmic spirals

Logarithmic Curvature Histogram (LCH)

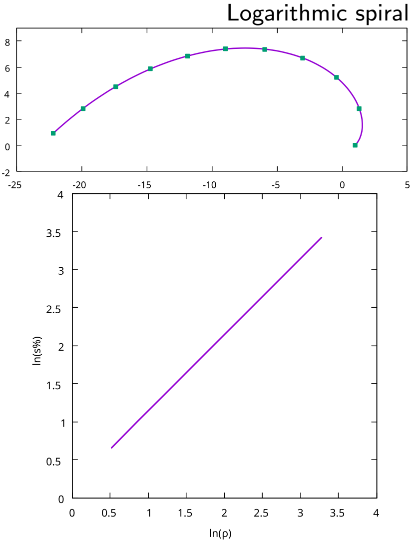
Curve shape evaluation:

1. Take samples of the curvature radius (ρ_i) at equal arc lengths
2. Divide $\ln(\rho_i)$ into a fixed number of bins
3. Plot the logarithm of the percentage of samples in the bins

$\rightarrow : \ln \rho$

$$\uparrow : \ln \frac{\partial s}{\partial \ln \rho} = \ln \frac{\partial s}{\partial \rho / \rho}$$

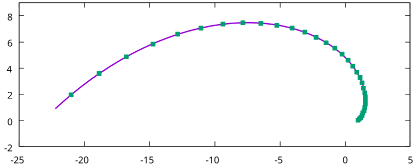
Straight lines are favorable

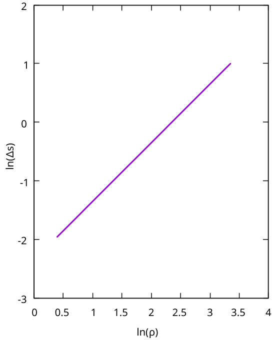


T. Harada, F. Yoshimoto, M. Moriyama: *An aesthetic curve in the field of industrial design*. Proceedings of IEEE Symposium on Visual Language, pp. 38–47, 1999.

LCH—Alternative Interpretation

Logarithmic spiral





1. Divide the curve into segments with the same $\Delta\rho/\rho$ ratio
2. Draw the log–log plot of segment lengths, i.e., $\ln(\Delta s)$ over $\ln(\rho)$

Linearity means

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

where α is the slope

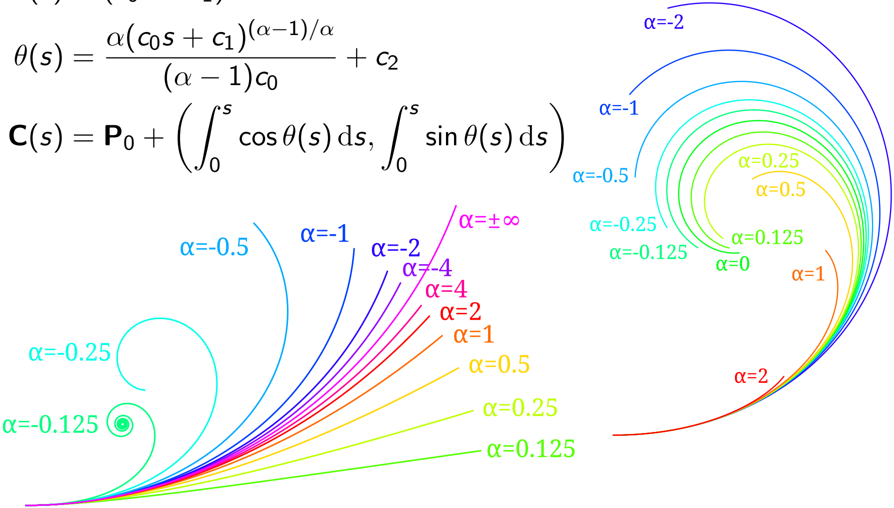
N. Yoshida, T. Saito: *Interactive aesthetic curve segments*. TVC 22(9-11):896–905, 2006.

Log-Aesthetic Curves

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

$$\theta(s) = \frac{\alpha(c_0 s + c_1)^{(\alpha-1)/\alpha}}{(\alpha-1)c_0} + c_2$$

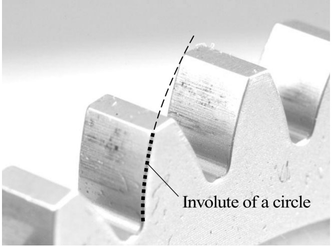
$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s\right)$$



K. T. Miura: *A general equation of aesthetic curves and its self-affinity*. CAD&A(1–4):457–464, 2006.

Types of Log-Aesthetic Curves

- Circle ($\alpha = \infty$ or $c_0 = 0$)
- Circle involute ($\alpha = 2$)
- Logarithmic spiral ($\alpha = 1$)
  - $\theta(s) = \ln(c_0 s + c_1)/c_0 + c_2$
- Nielsen's spiral ($\alpha = 0$)
  - $\kappa(s) = \exp(c_0 s + c_1)$
  - $\theta(s) = \exp(c_0 s + c_1)/c_0 + c_2$
- Clothoid ($\alpha = -1$)



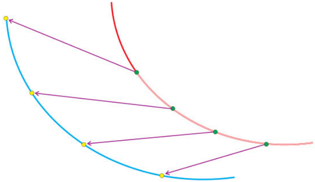
S. Radzevich: *Principal accomplishments in the scientific theory of gearing*. MATEC Web of Conferences 287, 2019.


Roller coaster


Nautilus shell

Properties

- Self-affinity
  - Weaker than self-similarity
  - The “tail” of a log-aesthetic arc can be affinely transformed into the whole curve
- Natural shape
  - Egg contour, butterfly wings, etc.
- Also appears in art and design
  - Japanese swords, car bodies, etc.



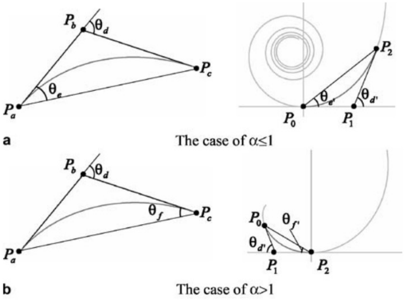
Scaling a segment shows self-affinity

A japanese sword

A swallowtail butterfly

Interpolation ★

- Input: 3 control points $\mathbf{P}_a$, $\mathbf{P}_b$, $\mathbf{P}_c$ (as before), α fixed
- Idea: find a segment of the curve in standard form
  - $\mathbf{P}_0 = 0$, $\theta(0) = 0$, $\kappa(0) = 1$
  - Transform the control points to match a segment



N. Yoshida, T. Saito: *Interactive aesthetic curve segments*. TVC 22(9-11):896–905, 2006.

Interpolation (2)

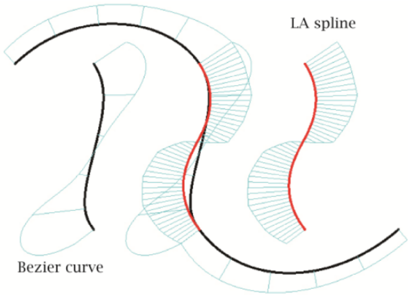
- In this form, the curve is defined by a scalar Λ:
  - $c_0 = \alpha\Lambda$, $c_1 = 1$, $c_2 = \frac{1}{(\alpha-1)\Lambda}$
- P_0, P_1 and P_2 are "points" on the complex plane
- P_0 is the origin, P_2 corresponds to $\mathbf{C}(s_0)$
  - s_0: total length (computed from θ_d)
- P_1 is found by intersection:

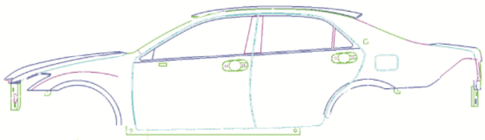
$$P_1 = \mathrm{Re}\left[P_2 + e^{i\theta_d} \cdot \left(-\frac{\mathrm{Im}(P_2)}{\mathrm{Im}(e^{i\theta_d})}\right)\right]$$

- The input triangle and transformed triangle should be similar
  - Find the value of Λ by iterative bisection
  - For $\alpha = 1$, Λ can be arbitrarily large (open-ended bisection)
  - Otherwise $\Lambda \in [0, \theta_d/(1-\alpha)]$
- Quite a few corner cases...

G^2 LA spline

- 3-segment spline, connecting with G^2 continuity
- Input: position, tangent & curvature at the endpoints
- Iterative; uses a Bézier curve to estimate total arc length
- Capable of S-shapes

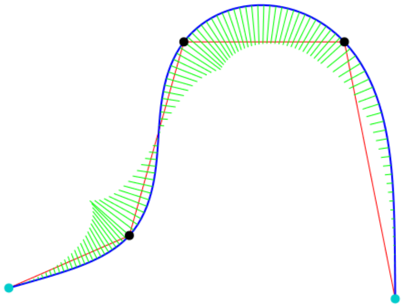




K. T. Miura, D. Shibuya, R. U. Gobithaasan, Sh. Usuki:
Designing log-aesthetic splines with G^2 continuity. CAD&A 10(6):1021–1032, 2013.

Discrete spline interpolation ★

- Input:
  - Points to interpolate
- Output:
  - Discrete curve (polygon)
  - Open or closed
  - Input points are knots (segment boundaries)
  - Each segment is LA, connected with G^2
- Originally for clothoids, but easily adapted to LAC



R. Schneider, L. Kobbelt: *Discrete fairing of curves and surfaces based on linear curvature distribution*. Technical report, Max Planck Institut für Informatik, Saarbrücken, 2000.

Discrete spline interpolation (2) – Algorithm

1. Subsample the input $\rightarrow \mathbf{Q}_i^0$
2. Compute discrete curvatures at input points:

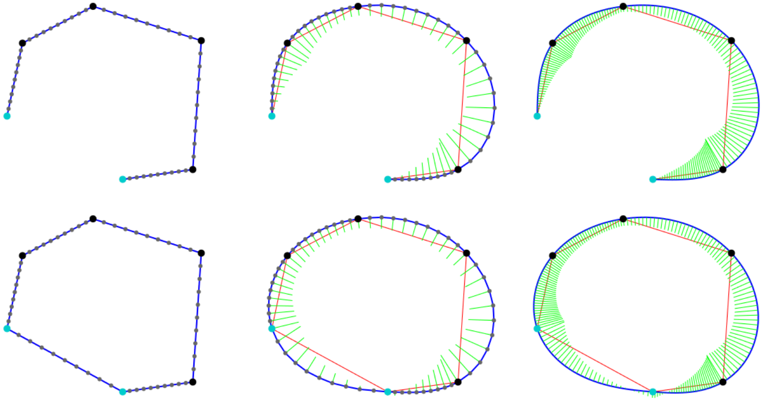
$$\kappa_i = 2\frac{\det(\mathbf{Q}_i^k - \mathbf{Q}_{i-1}^k, \mathbf{Q}_{i+1}^k - \mathbf{Q}_i^k)}{\left\|\mathbf{Q}_i^k - \mathbf{Q}_{i-1}^k\right\| \left\|\mathbf{Q}_{i+1}^k - \mathbf{Q}_i^k\right\| \left\|\mathbf{Q}_{i+1}^k - \mathbf{Q}_{i-1}^k\right\|}$$

3. Assign target curvatures to non-input points (based on α)
4. Compute new position of non-input points
   4.1 Local discrete curvature equals target curvature
   4.2 Segments are arc-length parameterized:

$$\left\|\mathbf{Q}_i^{k+1} - \mathbf{Q}_{i-1}^k\right\| = \left\|\mathbf{Q}_{i+1}^k - \mathbf{Q}_i^{k+1}\right\|$$

5. Back to step 2 (unless change was $< \varepsilon$ or too many iterations)

Discrete spline interpolation (3) – Example



(a) Subsampled input. (b) Output curve. (c) Dense output curve.

Outline

Generalized Catenaries

$$\kappa(s) = (c_0 s^2 + c_1 s + c_2)^{-1/\alpha}$$

- Generalization of LA curves (LA when $c_0 = 0$ or $c_1 = 2\sqrt{c_0 c_2}$)
- Includes catenaries: $\alpha = 1$, $c_0 = 1/a$, $c_1 = 0$, $c_2 = a$

$$\kappa(s) = \frac{a}{s^2 + a^2}, \quad \theta(s) = \arctan(s/a) + c, \quad y = a\cosh(x/a)$$

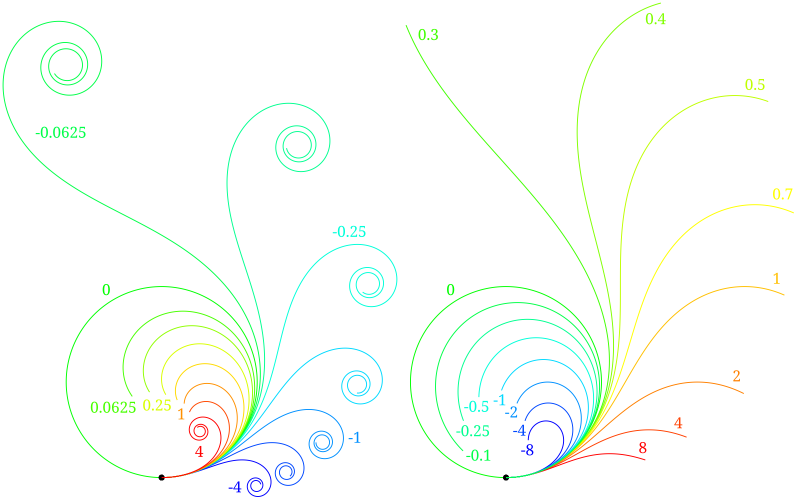
- 'Hyperbolic–elastic' subfamily: $\alpha = -1$, $c_1 = 0$

$$\kappa(s) = c \cdot s^2 + 1, \quad \theta(s) = \frac{1}{3} c \cdot s^3 + s$$

  - $c > 0$: resembles hyperbolic spirals
  - $c < 0$: starts off similarly to elastica

P. Salvi: *Generalized catenaries and trig-aesthetic curves*. CAD&A 23(1):56–67, 2026.

Generalized Catenaries ($\alpha = -1$) vs. Elastica

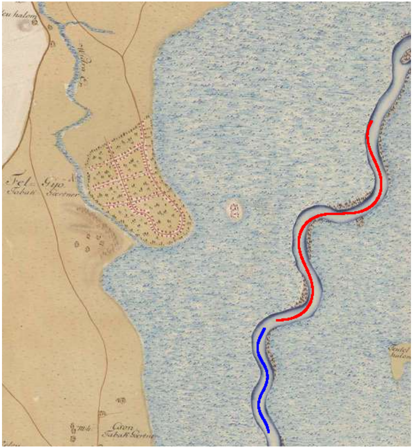


Trig-Aesthetic Curves

$$\kappa(s) = c_0 \cos(c_1 s + c_2), \quad \theta(s) = \frac{c_0}{c_1} \sin(c_1 s + c_2) + c_3$$

- 'Sine-generated curves'
- Used in geophysics (models river meandering)
- c_0: scaling
- c_1: **shape**
- c_2: starting parameter
- c_3: starting tangent
- Simpler version:

$$\kappa(s) = \cos(s/c)$$
$$\theta(s) = c \sin(s/c)$$



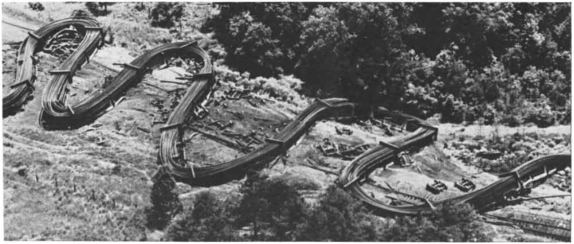
Meanders of the Tisza river (XIX. c.)

P. Salvi: *Generalized catenaries and trig-aesthetic curves.* CAD&A 23(1):56–67, 2026.

Connection with Elastica

$$\kappa(s) = \cos(s/c), \quad \theta(s) = c\sin(s/c)$$

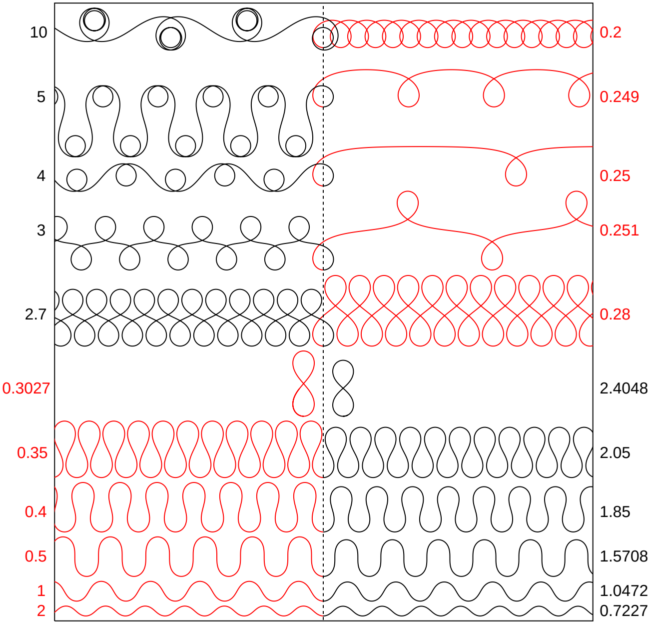
- Rivers meander along elastic curves
  - Most probable path of a particle turning by normal distribution
  - Minimize bending energy with fixed arc length
  - Solutions of $\theta''(s) + \lambda \sin\theta(s) = 0$
  - Maximum turning angle: $\arccos(1 - \frac{1}{2\lambda})$
- Trig-aesthetic curves are similar
  - Maximum turning angle: c



Wreck of a Southern Railway freight train near Greenville, S.C., 1965.

H. von Schelling: *Most frequent particle paths in a plane*. Eos 32(2):222–226, 1951.
W. B. Langbein, L. B. Leopold: *River meanders—Theory of minimum variance*. Technical Report 422-H, United States Geological Survey, 1966.

Trig-Aesthetic Curves vs. Elastica



Connection with Nielsen's Spiral

- Nielsen's Spiral (LA curve with $\alpha = 0$, $c_0 = 1/c$, $c_2 = 0$):

$$\theta_N(s) = c\exp(s/c + c_1),$$
$$\theta'_N(s) = \kappa(s) = \exp(s/c + c_1)$$

  Differential equation form:

$$\theta''_N(s) - \theta_N(s)/c^2 = 0$$

- Trig-aesthetic curve:

$$\theta(s) = c\sin(s/c), \quad \theta(s)'' = \kappa(s) = \cos(s/c)$$

  Differential equation form:

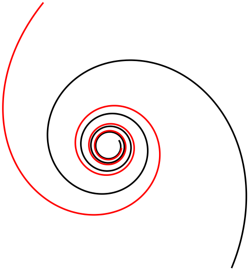
$$\theta''(s) + \theta(s)/c^2 = 0$$

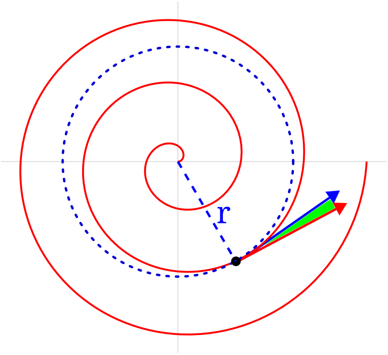
- Only the sign is different
  - Same when $c = -i$ (but initial values differ)

Connection with Hyperbolic Spiral ($c = -i$)

$$\kappa(s) = \cos(-s/i) = \cosh(s), \quad \theta(s) = \sinh(s)$$

- Pitch angle: angle between tangents to the spiral and a circle with the same center
- Hyperbolic spiral: pitch proportional to radius
- TA curve with $c = -i$: pitch converges to radius



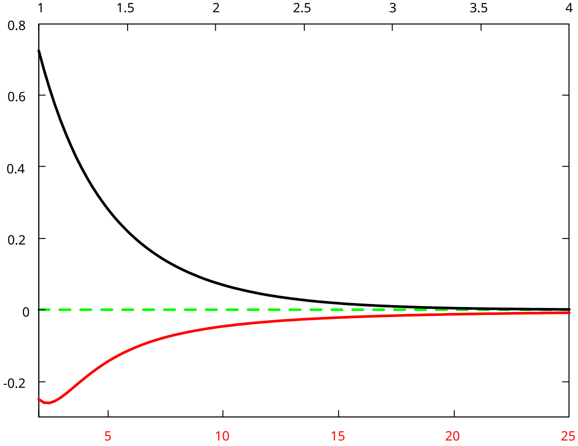


(Arithmetic spiral)

Connection with Hyperbolic Spiral ($c = -i$)

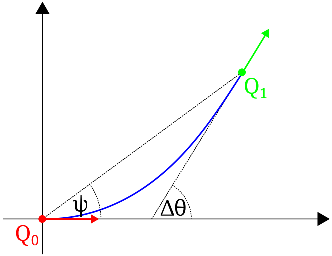
Comparison of the LCH slope function

$$\alpha(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right) = 1 - \frac{\rho(s)\rho''(s)}{\rho'(s)^2}$$



Hermite Interpolation

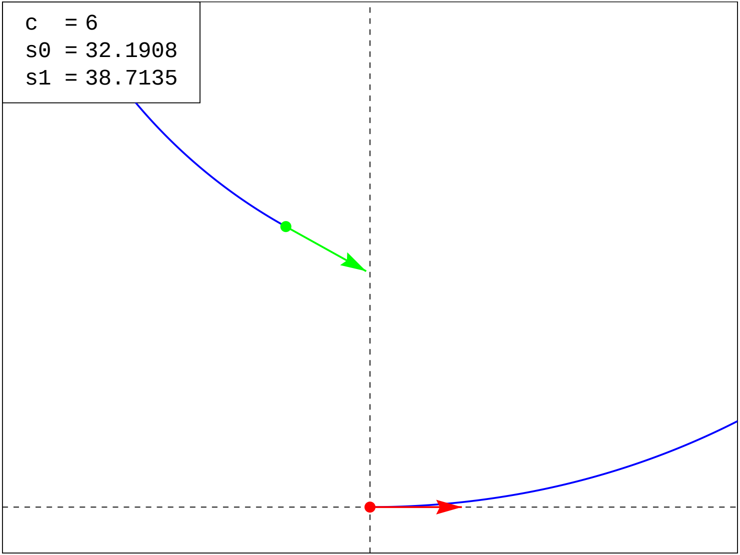
- Similarly to log-aesthetic curves
- Translation, rotation, scaling $\rightarrow$ irrelevant
- Simplified problem: two constraints (ψ and $\Delta\theta$)



- Variables: $[s_0, s_1]$ interval (c fixed)
- If we know $s_0 \Rightarrow$ we can compute s_1
- Determine s_0 by binary search
- Initial bracket by sampling

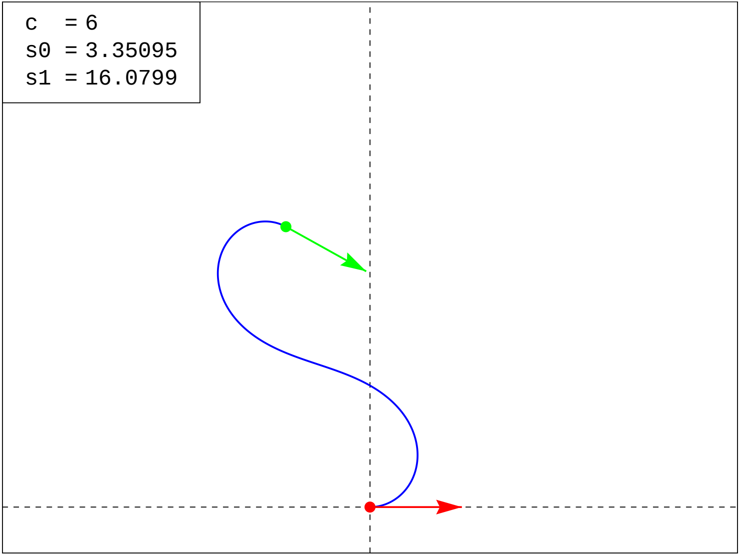
Hermite Interpolation—Choosing a Solution

Multiple solutions, some inferior



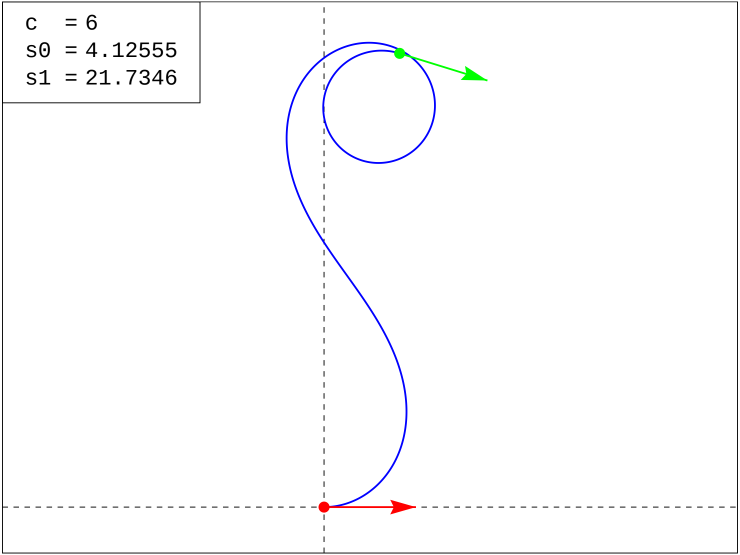
Hermite Interpolation—Choosing a Solution

Minimize arc length $E_s = \|\mathbf{Q}_1 - \mathbf{Q}_0\|(s_1 - s_0)/\|\mathbf{C}(s_1) - \mathbf{C}(s_0)\|$



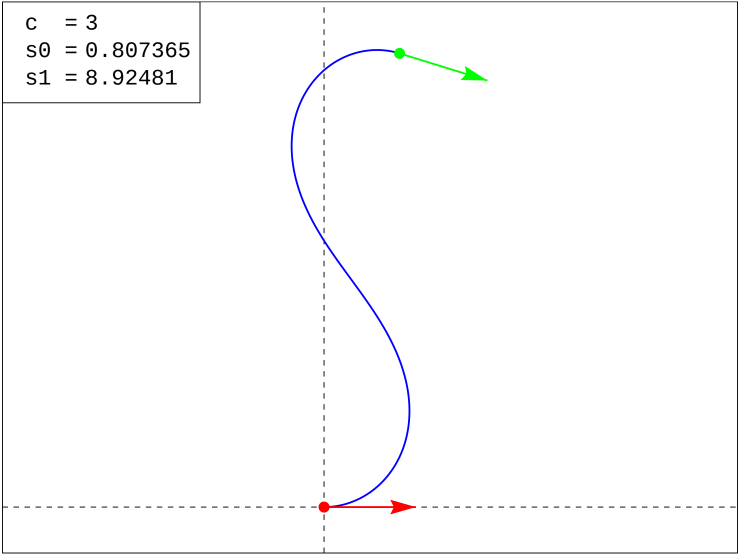
Hermite Interpolation—Choosing the Shape Parameter

Large c may result in loops



Hermite Interpolation—Choosing the Shape Parameter

Choose smaller c (but $c \geq |\Delta\theta|$)

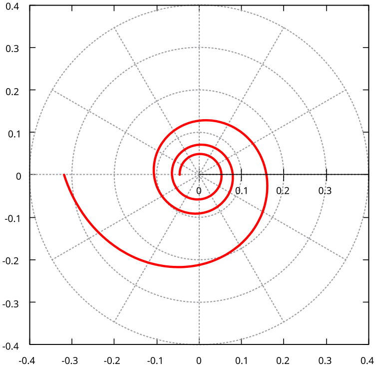


Outline

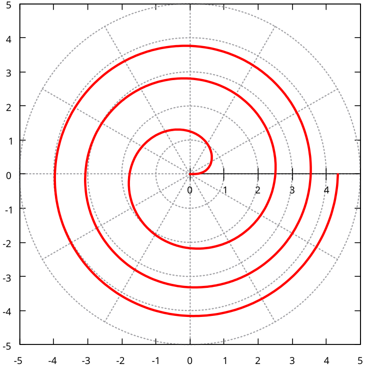
Generalized Archimedean Spirals

Polar equation: $r = a + b\phi^{\frac{1}{c}}$

- $c = -2$: lituus
- $c = -1$: hyperbolic spiral
- $c = 1$: Archimedean (arithmetic) spiral
- $c = 2$: Fermat's spiral



Hyperbolic spiral



Fermat's spiral

Radial Curves

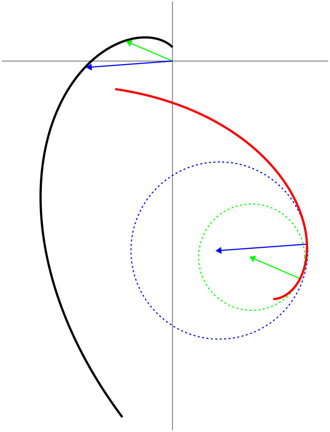
- Vector to the center of curvature, placed at the origin
- $\theta(t)$: tangent angle to the x axis
- $\theta^{\perp}(t) = \theta(t) + \frac{\pi}{2}$
- $\mathbf{R}(t) = [\cos\theta^{\perp}(t), \sin\theta^{\perp}(t)] \cdot \rho(t)$
- For log-aesthetic curves:

$$\rho(\theta^{\perp}) = \left(\theta^{\perp} c_0 \frac{\alpha - 1}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

- Polar equation:

$$r = b\phi^{\frac{1}{\alpha-1}}$$

- GA spiral with $a = 0$ and $c = \alpha - 1$



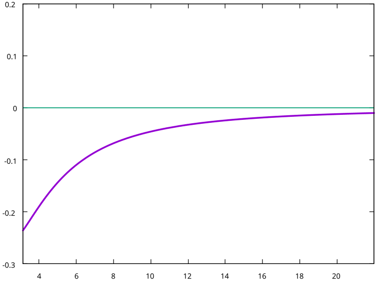
Logarithmic spiral
(special case: $r = e^{b\phi}$)

P. Salvi: *Log-aesthetic curves and generalized Archimedean spirals*. CAGD 121:102468, 2025.

LCH Slope of GA spirals with $a = 0$

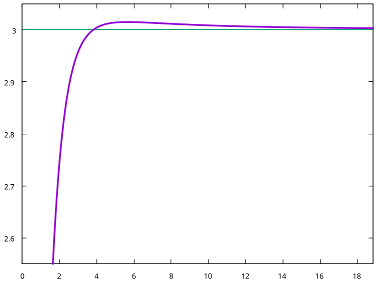
$$\alpha(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right)$$

Approaches $c + 1$ (slope of the related LA curve)



$c = -1$ (Hyperbolic spiral)

$\rightarrow \alpha = 0$ (Nielsen's spiral)



$c = 2$ (Fermat's spiral)

$\rightarrow \alpha = 3$

Approximating LA curves by GA spirals

- LA curve segment:

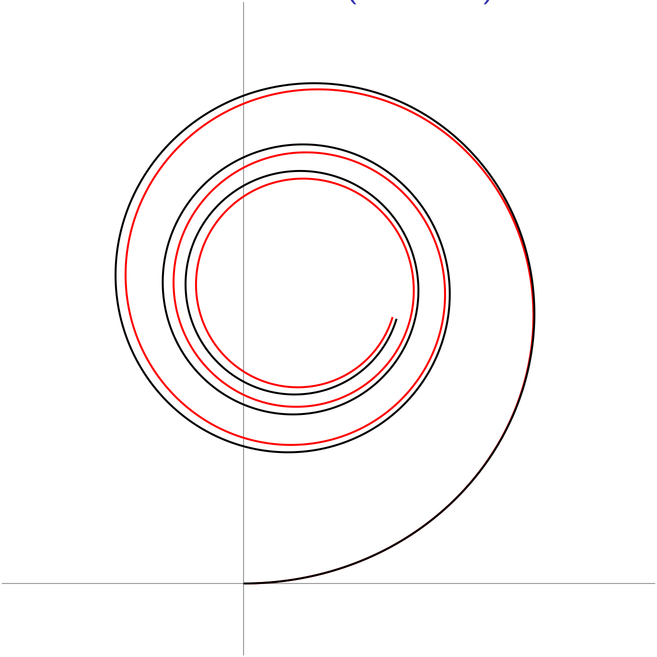
$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s \right), \quad s \in [s_{\min}, s_{\max}]$$

- GA spiral segment:

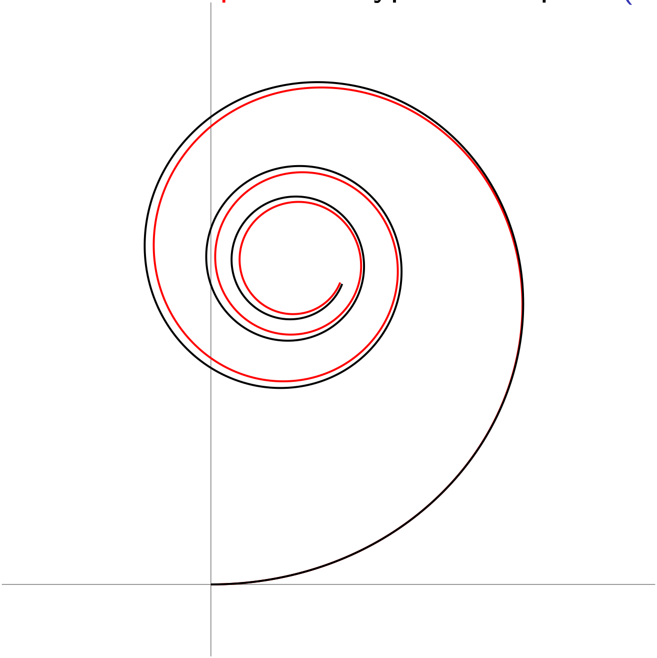
$$\mathbf{C}_{\mathrm{GA}}(t) = [\cos t, \sin t] \cdot (a + bt^{\frac{1}{c}}), \quad t \in [t_{\min}, t_{\max}]$$

- $a = 0$ and $c = \alpha - 1$
- Assume matching starting point and direction
  - Simple translation/rotation
- Interpolate curvature at $t_{\min}$
  - If $t_{\min}$ is known $\rightarrow$ b can be computed
- Interpolate curvature derivative at $t_{\min}$
  - $t_{\min}$ found by binary search
  - Initial frame by iterative doubling

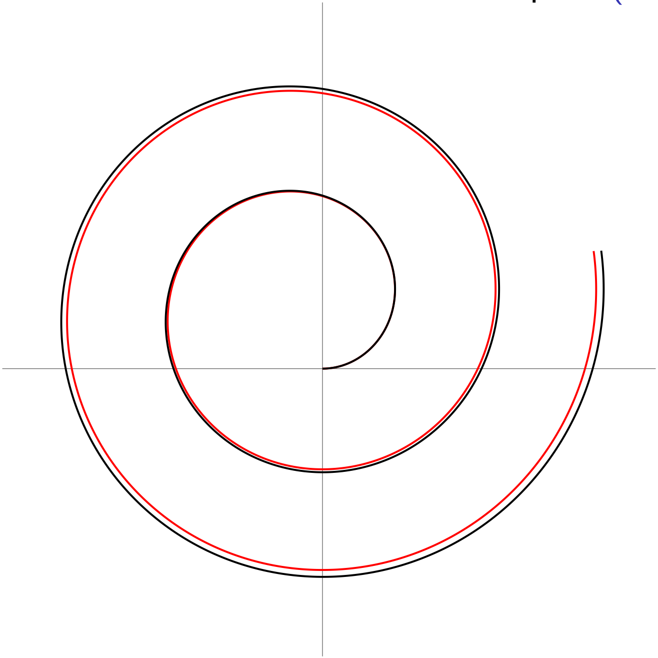
Example 1: clothoid vs. lituus ($\alpha = -1$)



Example 2: Nielsen's spiral vs. hyperbolic spiral $(\alpha = 0)$



Example 3: Circle involute vs. arithmetic spiral ($\alpha = 2$)



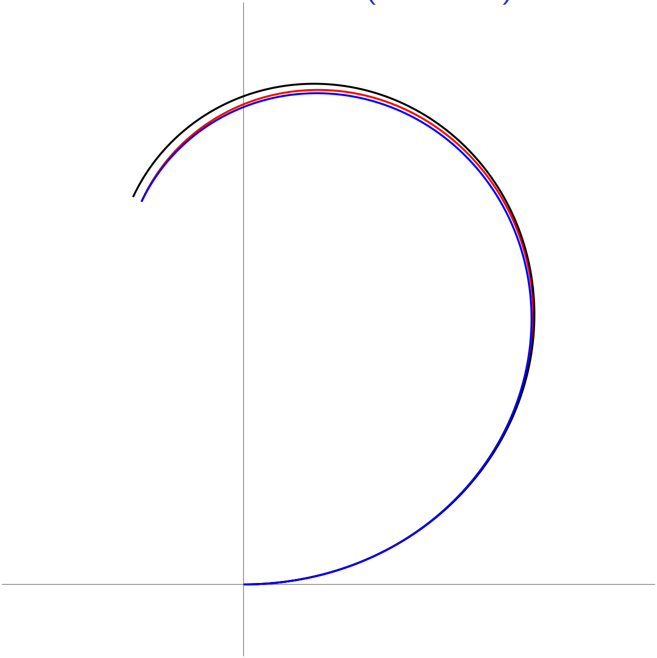
Alternative Constraint

- Idea: Fix the endpoint instead of the curvature derivative
- Different error function for the bisection search
  - Radial distance of the endpoint to the GA spiral

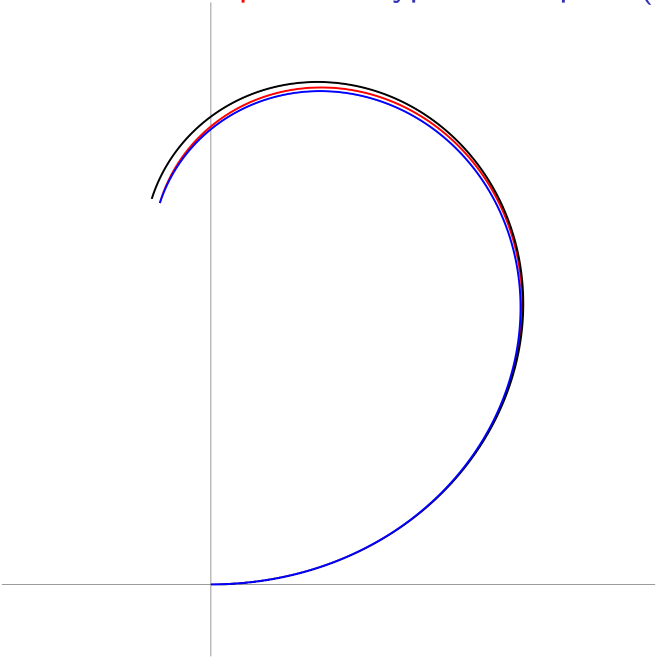
Algorithm

1. Rotate the spiral s.t. $\mathbf{C}'_{\mathrm{GA}}(t_{\mathsf{min}})$ points to $\theta(s_{\mathsf{min}})$.
2. Set $\mathbf{Q}$ (the spiral center) s.t. $\mathbf{Q} + \mathbf{C}_{\mathrm{GA}}(t_{\mathsf{min}}) = \mathbf{P}_0$.
3. Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors from $\mathbf{Q}$ to $\mathbf{P}_0$ and $\mathbf{C}(s_{\mathsf{max}})$.
4. Set $t_{\mathsf{max}} = t_{\mathsf{min}} + \arccos\langle \mathbf{u}, \mathbf{v} \rangle$, or, if $\det(\mathbf{u}, \mathbf{v}) < 0$, choose the larger angle: $t_{\mathsf{max}} = t_{\mathsf{min}} + 2\pi - \arccos\langle \mathbf{u}, \mathbf{v} \rangle$.
5. The error is $\|\mathbf{C}(s_{\mathsf{max}}) - \mathbf{Q}\| - \|\mathbf{C}_{\mathrm{GA}}(t_{\mathsf{max}})\|$.

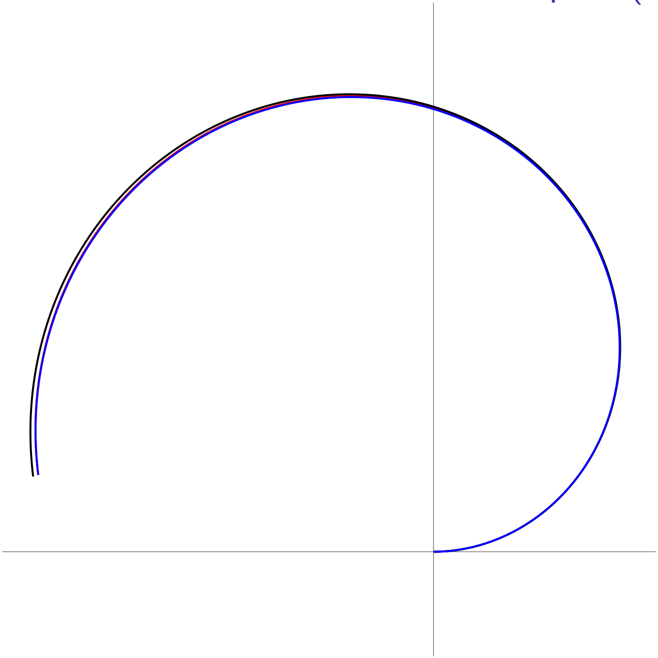
Example 1b: clothoid vs. lituus ($\alpha = -1$)



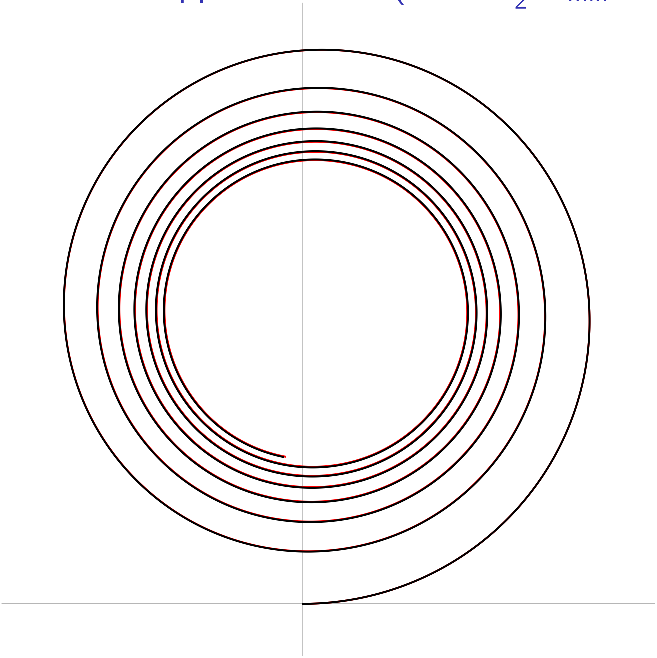
Example 2b: Nielsen's spiral vs. hyperbolic spiral ($\alpha = 0$)



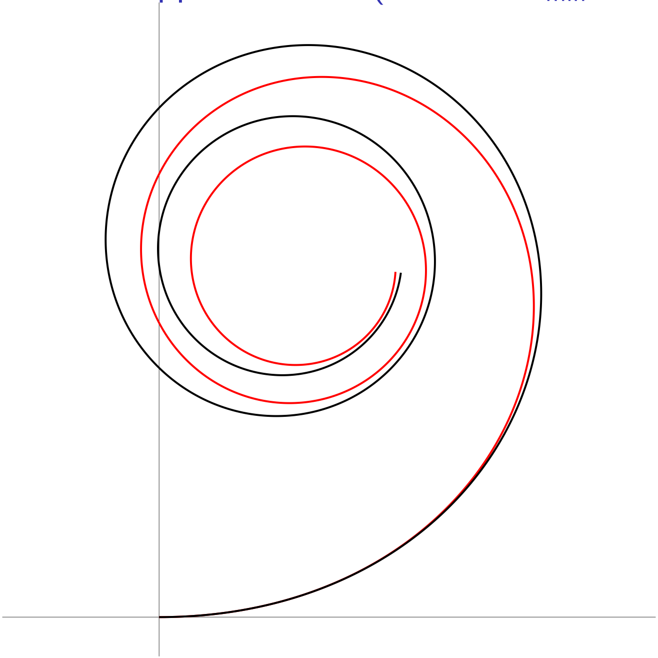
Example 3b: Circle involute vs. arithmetic spiral ($\alpha = 2$)



Example 4: Good Approximation ($\alpha = -\frac{3}{2}$, $t_{\min} \approx 9.88$)



Example 5: Bad Approximation ($\alpha = -1$, $t_{\min} \approx 1.42$)



Reconstructing Log-Aesthetic Curves from Radials

- From the construction: $\|\mathbf{C}'(t)\| = \|\mathbf{R}(t)\|$
- Inverse radial:

$$\mathbf{C}(t) = \int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \mathbf{R}(t)\,\mathrm{d}t$$

- Explicit equations for some cases, e.g.:
  - $b = 1$, $c = 1$ (circle involute):

$$[t\cos t - \sin t, t\sin t + \cos t]$$

  - $b = 1$, $c = \frac{1}{2}$:

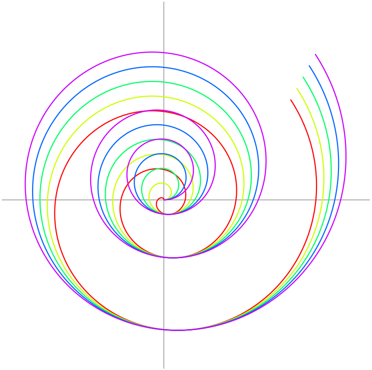
$$[(t^2 - 2)\cos t - 2t\sin t, (t^2 - 2)\sin t + 2t\cos t]$$

  - etc.
- May involve incomplete gamma functions

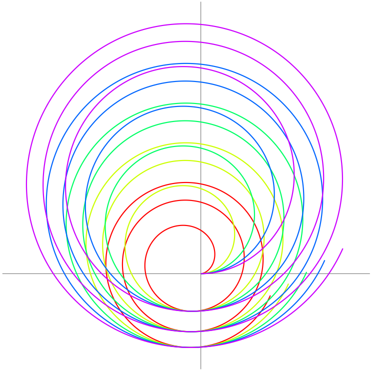
Generalized Log-Aesthetic Curves

What if $a \neq 0$?

- Arithmetic spirals ($c = 1$): just a shift
- $c < 0$: LCH slope diverges into $\pm\infty$
- $c > 0$: Still converges to $c + 1$



$c = \frac{1}{2}$, $a \in \{0, 20, 40, 60, 80\}$



$c = 2$, $a \in \{0, 1, 2, 3, 4\}$

Outline

Cesàro's Invariants

- Series of radii of curvature:

$$\rho_{(0)} = \rho, \qquad \rho_{(k)} = \rho\rho'_{(k-1)}$$

- $\rho_{(k)} = \frac{\mathrm{d}\rho_{(k-1)}}{\mathrm{d}\theta}$, all radii are of the same scale
- Invariant: $f(\rho, \rho_{(1)}, \ldots, \rho_{(k)}) \equiv 0$
- Example: parabola with stretch a (e.g. $y = ax^2 + bx + c$)
  - $\rho = \frac{\sec^3\theta}{2a} \quad \Rightarrow \quad 9\rho^2 + 4\rho_{(1)}^2 - 3\rho\rho_{(2)} \equiv 0$
- Proposition: use $f(\rho, \rho_{(1)}, \ldots, \rho_{(k)}) \equiv \mathrm{const.}$
  - More information (eliminate only non-shape parameters)
  - Often more concise expressions
  - E.g. parabola: $(\rho_{(1)}^2/\rho^2 + 9)^3/\rho^2 \equiv (54a)^2$
- Better for aesthetic curves: κ, κ', κ'', ...
- ODE form: $\theta'' = f(\theta, \theta') \Rightarrow \kappa' = f(\theta, \kappa)$, useful for plotting

E. Cesàro: *Lezioni di geometria intrinseca*. Naples, 1896.

Table of Invariants

	Elastica
Intrinsic	$\mathrm{cn}(\sqrt{\lambda}s, \frac{1}{4\lambda})$
ODE	$-\lambda \sin\theta$
Constant	$\kappa'^2 + \kappa''^2/\kappa^2 = \lambda^2$
Invariant	$\kappa\kappa''' + \kappa'(\kappa^3 - \kappa'')$

	Log-Aesthetic Curves ($\alpha \neq 0$)	Nielsen's spiral ($\alpha = 0$)
Intrinsic	$(s+1)^{-\frac{1}{\alpha}}$	$\exp(s)$
ODE	$-\kappa^{\alpha+1}/\alpha$	κ
Constant	$\kappa\kappa''/\kappa'^2 = \alpha + 1$	N/A
Invariant	$\kappa'^2\kappa'' + \kappa\kappa'\kappa''' - 2\kappa\kappa''^2$	$\kappa - \kappa'$

	Trig-Aesthetic Curves	Complex TAC
Intrinsic	$\cos(s/c)$	$\cosh(s/c)$
ODE	$-\theta/c^2$	θ/c^2
Constant	$(1-\kappa^2)/\kappa'^2 = c^2$	$\Leftarrow$
Invariant	$\kappa\kappa'^2 + \kappa''(1-\kappa^2)$	$\Leftarrow$

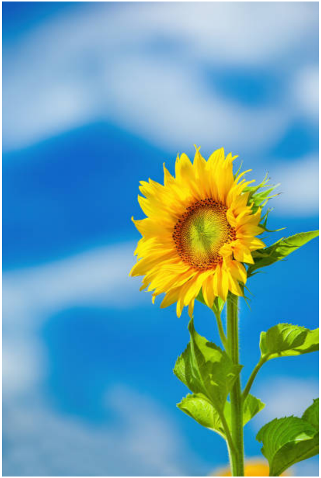
Circle / Clothoid / Nielsen's spiral / TAC common constant form: κ''/κ
LAC–TAC common constant form: $\kappa\kappa'''/\kappa'\kappa'' = 2\alpha + 1$ (Nielsen $\approx$ TAC)

P. Salvi: *A note on invariants of aesthetic curve families.* (in preparation)

Outline

Conclusion

- Log-aesthetic curve family
  - Generalizes classical curves
  - 3-point interpolation
  - Discrete spline
- Generalizations
  - Generalized catenaries
  - Trig-aesthetic curves
  - $\Leftrightarrow$ Hyperbolic/Nielsen's spiral
  - $\Leftrightarrow$ Elastica
- $\Leftrightarrow$ Archimedean spirals
  - Approximation
- Invariants of curve families



Spirals of sunflower seeds

https://3dgeo.iit.bme.hu/