On the CAD-compatible conversion of S-patches

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Outline

1. Introduction
   - Motivation
   - Previous work

2. Simplexes & S-patches
   - Simplexes
   - S-patches

3. Conversion
   - Conversion to quadrilateral S-patch
   - Conversion to tensor product form

4. Conclusion
   - Example
   - Discussion
Multi-sided surfaces in CAD software

- Standard surface representations:
  - Tensor-product Bézier surface
  - Tensor-product B-spline surface
  - Tensor-product NURBS surface
- No standard multi-sided representation
- Conversion to tensor-product patches
  - Trimming
    - Parameterization issues
    - Asymmetric
    - Not watertight
  - Central split
    - Loosely defined dividing curves
    - Only $C^0$ or $G^1$ continuity
Solution

- **Exact tensor product conversion**
- Trimmed rational Bézier surface
  - Only polynomial (Bézier) boundaries
  - Trimming curves ⇒ lines in the domain
- Native $n$-sided representation
  - S-patch
  - Generalization of Bézier curves & triangles
  - Suitable for $G^1$ hole filling [1]


In: Proceedings of the 12th Conference of the Hungarian Association for Image Processing and Pattern Recognition, 2019 (accepted).
S-patches & simplexes

- [1989, Loop & DeRose]
  A multi-sided generalization of Bézier surfaces
  - The original S-patch publication
  - Contains *theoretical results* on the tensor product conversion
  - Missing from the description of the algorithm:
    - Composition of rational Bézier simplexes
    - Blossom of Wachspress coordinates

- [1987, Ramshaw]
  Blossoming: A connect-the-dots approach to splines

- [1988, DeRose]
  Composing Bézier simplexes

- [1993, DeRose et al.]
  Functional composition algorithms via blossoming
Simplex in $nD$

- $(n + 1)$ points in $nD$
- Let $V_i$ denote these points
- Any $nD$ point is uniquely expressed by the affine combination of $V_i$:
  
  $$p = \sum_{i=1}^{n} \lambda_i V_i \quad \text{with} \quad \sum_{i=1}^{n} \lambda_i = 1$$

- $\lambda_i$ are the barycentric coordinates of $p$ relative to the simplex

(images from Wikipedia)
Bézier curve

Let's look at the equation of a Bézier curve:

\[ C(u) = \sum_{i=0}^{d} P_i B_i^d(u) \]
Bézier curve

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Let \( s = (i, d - i) \) and \( \lambda = (u, 1 - u) \).
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\]

Let \( s = (i, d - i) \) and \( \lambda = (u, 1 - u) \).

Then

\[
C(\lambda) = \sum_{s} P_s \frac{d!}{s_1!s_2!} \lambda_1^{s_1} \lambda_2^{s_2}
\]
Bézier triangle

Now let’s look at the equation of a Bézier triangle:

\[ T(\lambda) = \sum_{s} P_s \frac{d!}{s_1!s_2!s_3!} \lambda_1^{s_1} \lambda_2^{s_2} \lambda_3^{s_3} = \sum_{s} P_s B_s^d(\lambda) \]

- \( s = (s_1, s_2, s_3) \) with \( s_i \geq 0 \) and \( s_1 + s_2 + s_3 = d \)
- \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) barycentric coordinates of a 2D point relative to the domain triangle (simplex)

Did you know?

This was *Paul de Casteljau’s* generalization of Bézier curves.
- “Bézier” curves were also his invention
- Tensor product surfaces were invented by *Pierre Bézier*
- de Casteljau worked at Citroën, while Bézier at Renault
Bézier simplex

- The logical generalization to \((n-1)\) dimensions:

\[
S(\lambda) = \sum_{s} P_{s} \frac{d!}{\prod_{i=1}^{n} s_{i}!} \prod_{i=1}^{n} \lambda_{i}^{s_{i}} = \sum_{s} P_{s} B_{s}^{d}(\lambda)
\]

- \(s = (s_1, s_2, \ldots, s_n)\) with \(s_i \geq 0\) and \(\sum_{i=1}^{n} s_i = d\)

- \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) barycentric coordinates of an \((n-1)D\) point relative to the domain simplex

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**Note**

Bézier simplexes are mappings, not geometric entities!
S-patches as Bézier simplexxes

- S-patch equation \((n \text{ sides, depth } d)\):
  \[
  S(\lambda) = \sum_s P_s \frac{d!}{\prod_{i=1}^n s_i!} \prod_{i=1}^n \lambda_i^{s_i} = \sum_s P_s B_s^d(\lambda)
  \]

- Domain for an \(n\)-sided S-patch:
  - Regular \(n\)-sided polygon (in 2D)
- Domain for an \((n - 1)\)-dimensional Bézier simplex:
  - An \((n - 1)\)-dimensional simplex (\(n\) barycentric coordinates)
- Needed:
  - Mapping from an \(n\)-sided polygon to \(n\) barycentric coordinates
  - Generalized barycentric coordinates
    - E.g. Wachspress, mean value, etc.
  - Defines an embedding in the \((n - 1)\)-dimensional simplex
S-patches

Control structure

- Very complex – many control points, hard to use manually
- Boundary control points define degree $d$ Bézier curves
- Adjacent control points have shifted labels, e.g. $21000 \rightarrow 30000, 11001, 20100, 12000$
Overview

Claim 6.4 in [1989, Loop & DeRose]
For every \( m \)-sided regular S-patch of depth \( d \), there exists an equivalent \( n \)-sided regular S-patch of depth \( d(m - 2) \).

Lemma 6.2 in [1989, Loop & DeRose]
For every 4-sided regular S-patch of depth \( d \), there exists an equivalent tensor product Bézier patch of degree \( d \).

1. Convert the \( n \)-sided S-patch of depth \( d \) to a quadrilateral S-patch of depth \( d(n - 2) \).
2. Convert the quadrilateral S-patch to a tensor product Bézier patch of degree \( d(n - 2) \).
Conversion to quadrilateral S-patch

Conversion as simplex composition

- Wachspress coordinates on an \( n \)-sided polygon
  - \( \ldots \) have a Bézier simplex form (denoted by \( W_n \))
  - \( \ldots \) are pseudoaffine (have an affine left inverse \( W_n^{-1} \))
- Mapping from the domain polygon to a 3D point:
  \[ S \circ W_n \]
Conversion as simplex composition

- Wachspress coordinates on an $n$-sided polygon
  - ... have a Bézier simplex form (denoted by $W_n$)
  - ... are pseudoaffine (have an affine left inverse $W_n^{-1}$)
- Mapping from the domain polygon to a 3D point:

$$S \circ W_n = S \circ W_n \circ (W_4^{-1} \circ W_4)$$
Conversion as simplex composition

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Conversion as simplex composition

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- The 4-sided formulation is the composition of 3 simplexes:
  - $W_4^{-1}$: defined by the vertices of the rectangular domain
  - $S$: the S-patch (with homogenized control points)
  - $W_n$: ??? [a rational Bézier simplex of degree $n - 2$]
- Composition:
  - Two algorithms (simple vs. efficient) [see the paper]
Determining the control points of $W_n$ – homogenization

$$\lambda_i(p) = \frac{\prod_{j \neq i-1, i} D_j(p)}{\sum_{k=1}^{n} \prod_{j \neq k-1, k} D_j(p)}$$

- $D_j(p)$ is the signed distance of $p$ from the $j$-th side
- Rational expression $\Rightarrow$ homogenized coordinates
  - Use the barycentric coordinates as “normal” coordinates
  - $(x, y, z) \equiv (wx, wy, wz, w(1-x-y-z))$
- Homogenized form of $W_n$:

$$\left\{ \prod_{j \neq i-1, i} D_j(p) \right\}$$
Determining the control points of $W_n$ – polarization

For any homogeneous polynomial $Q(u)$ of degree $d$, $\exists q$ s.t.

\[
q(u_1, \ldots, u_d) = q(u_{\pi_1}, \ldots, u_{\pi_d}),
\]

\[
q(u_1, \ldots, \alpha u_{k_1} + \beta u_{k_2}, \ldots, u_d) = \alpha q(u_1, \ldots, u_{k_1}, \ldots, u_d) + \beta q(u_1, \ldots, u_{k_2}, \ldots, u_d),
\]

\[
q(u, \ldots, u) = Q(u).
\]

Then $q$ is called the blossom of $Q$.

The control points of its Bézier simplex form are

\[
P_s^Q = q(V_1, \ldots, V_{s_1}, V_2, \ldots, V_{s_2}, \ldots, V_n, \ldots, V_{s_n}),
\]

where $V_i$ are the vertices of the simplex.
Determining the control points of $W_n$ – blossom

The blossom of $W_n$ is

$$q(p_1, \ldots, p_{n-2})_i = \frac{1}{(n-2)!} \cdot \sum_{\pi \in \Pi(n-2)} \prod_{k=1}^{n-2} D_j(p_{\pi_k})$$

- $\Pi(n-2)$ is the set of permutations of $\{1, \ldots, n-2\}$
- $k$ runs from 1 to $n-2$ while $j$ from 1 to $n$ skipping $i-1$ and $i$

With this, the control points can be computed

- Simplex composition gives the quadrilateral S-patch
- Convert to “normal” homogeneous coordinates $(wx, wy, wz, w)$
Conversion to tensor product form

Explicit formula for tensor product control points

An 4-sided S-patch of depth $d$ can be represented as

$$\hat{S}(u, v) = \sum_{i=0}^{d} \sum_{j=0}^{d} C_{ij} B_i^d(u) B_j^d(v),$$

where

$$C_{ij} = \sum_{\mathbf{s}} \frac{\binom{d}{s}}{\binom{d}{i} \binom{d}{j}} P_{\mathbf{s}}.$$
Converting a 5-sided patch – control net
Converting a 5-sided patch – contours
Converting a 5-sided patch – trimmed tensor product
Converting a 5-sided patch – untrimmed tensor product
Limitations

- Efficiency
  - \( n = 5, \ d = 8 \) took \( > 5 \) minutes on a modern machine
    (How long would it have taken in 1989?)
  - Much faster algorithm is developed (see our upcoming paper)

- 3-sided patches
  - For Bézier triangles, the resulting patch is not rational
  - But there are simple alternative methods, e.g. [1992, Warren]

- Control net quality
  - Singularities on a circle around the domain
    - Denominator of Wachspress coordinates vanishes
  - Unstable control points near the corners

- Conclusion
  - The algorithm works, but it is not practical
Any questions?

Thank you for your attention.