On the CAD-compatible conversion of S-patches

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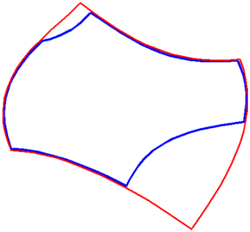
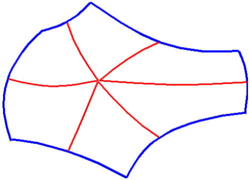
Outline

Multi-sided surfaces in CAD software

- Standard surface representations:
  - Tensor-product Bézier surface
  - Tensor-product B-spline surface
  - Tensor-product NURBS surface
- No standard multi-sided representation
- Conversion to tensor-product patches
  - Trimming
    - Parameterization issues
    - Asymmetric
    - Not watertight
  - Central split
    - Loosely defined dividing curves
    - Only C^0 or G^1 continuity





Solution

- Exact tensor product conversion
- Trimmed rational Bézier surface
  - Only polynomial (Bézier) boundaries
  - Trimming curves $\Rightarrow$ lines in the domain
- Native n-sided representation
  - S-patch
  - Generalization of Bézier curves & triangles
  - Suitable for G^1 hole filling [1]

[1] P. Salvi, *G^1 hole filling with S-patches made easy*.
In: Proceedings of the 12th Conference of the Hungarian Association for Image Processing and Pattern Recognition, 2019 (accepted).

S-patches & simplexes

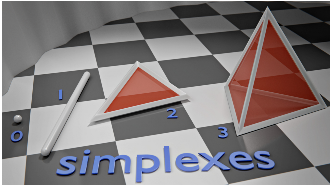
- [1989, Loop & DeRose]
  A multi-sided generalization of Bézier surfaces
  - The original S-patch publication
  - Contains *theoretical results* on the tensor product conversion
  - Missing from the description of the algorithm:
    - Composition of rational Bézier simplexes
    - Blossom of Wachspress coordinates
- [1987, Ramshaw]
  Blossoming: A connect-the-dots approach to splines
- [1988, DeRose]
  Composing Bézier simplexes
- [1993, DeRose et al.]
  Functional composition algorithms via blossoming

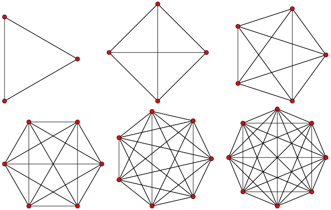
Simplex in nD

- $(n+1)$ points in nD
- Let V_i denote these points
- Any nD point is uniquely expressed by the affine combination of V_i:

$$p = \sum_{i=1}^{n} \lambda_i V_i \qquad \text{with} \sum_{i=1}^{n} \lambda_i = 1$$

- λ_i are the barycentric coordinates of p relative to the simplex





(images from Wikipedia)

Bézier curve

Let's look at the equation of a Bézier curve:

$$C(u) = \sum_{i=0}^{d} P_i B_i^d(u)$$

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Let $\mathbf{s} = (i, d-i)$ and $\lambda = (u, 1-u)$.
Then

$$C(\lambda) = \sum_{\mathbf{s}} P_{\mathbf{s}} \frac{d!}{s_1! s_2!} \lambda_1^{s_1} \lambda_2^{s_2}$$

Bézier triangle

Now let's look at the equation of a Bézier triangle:

$$T(\lambda) = \sum_{\mathbf{s}} P_{\mathbf{s}} \frac{d!}{s_1!s_2!s_3!} \lambda_1^{s_1} \lambda_2^{s_2} \lambda_3^{s_3} = \sum_{\mathbf{s}} P_{\mathbf{s}} B_{\mathbf{s}}^d(\lambda)$$

- $\mathbf{s} = (s_1, s_2, s_3)$ with $s_i \geq 0$ and $s_1 + s_2 + s_3 = d$
- $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ barycentric coordinates of a 2D point relative to the domain triangle (simplex)

Did you know?

This was *Paul de Casteljau*'s generalization of Bézier curves.

- "Bézier" curves were also his invention
- Tensor product surfaces were invented by *Pierre Bézier*
- de Casteljau worked at Citroën, while Bézier at Renault

Bézier simplex

- The logical generalization to $(n-1)$ dimensions:

$$S(\lambda) = \sum_{\mathbf{s}} P_{\mathbf{s}} \frac{d!}{\prod_{i=1}^{n} s_i!} \prod_{i=1}^{n} \lambda_i^{s_i} = \sum_{\mathbf{s}} P_{\mathbf{s}} B_{\mathbf{s}}^{d}(\lambda)$$

- $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ with $s_i \geq 0$ and $\sum_{i=1}^{n} s_i = d$
- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ barycentric coordinates of an $(n-1)$D point relative to the domain simplex

Note

Bézier simplexes are mappings, not geometric entities!

S-patches as Bézier simplexes

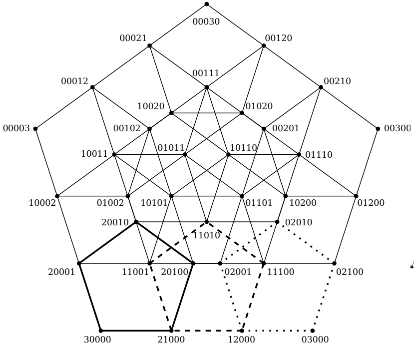
- S-patch equation (n sides, depth d):

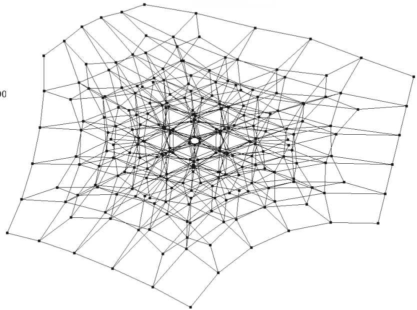
$$S(\lambda) = \sum_{\mathbf{s}} P_{\mathbf{s}} \frac{d!}{\prod_{i=1}^{n} s_i!} \prod_{i=1}^{n} \lambda_i^{s_i} = \sum_{\mathbf{s}} P_{\mathbf{s}} B_{\mathbf{s}}^{d}(\lambda)$$

- Domain for an n-sided S-patch:
  - Regular n-sided polygon (in 2D)
- Domain for an $(n-1)$-dimensional Bézier simplex:
  - An $(n-1)$-dimensional simplex (n barycentric coordinates)
- Needed:
  - Mapping from an n-sided polygon to n barycentric coordinates
  - Generalized barycentric coordinates
    - E.g. Wachspress, mean value, etc.
  - Defines an embedding in the $(n-1)$-dimensional simplex

Control structure

- Very complex – many control points, hard to use manually
- Boundary control points define degree d Bézier curves
- Adjacent control points have shifted labels,
  e.g. 21000 $\rightarrow$ 30000, 11001, 20100, 12000





Overview

Claim 6.4 in [1989, Loop & DeRose]

For every m-sided regular S-patch of depth d, there exists an equivalent n-sided regular S-patch of depth $d(m-2)$.

Lemma 6.2 in [1989, Loop & DeRose]

For every 4-sided regular S-patch of depth d, there exists an equivalent tensor product Bézier patch of degree d.

1. Convert the n-sided S-patch of depth d to a quadrilateral S-patch of depth $d(n-2)$.
2. Convert the quadrilateral S-patch to a tensor product Bézier patch of degree $d(n-2)$.

Conversion as simplex composition

- Wachspress coordinates on an n-sided polygon
  - ... have a Bézier simplex form (denoted by W_n)
  - ... are *pseudoaffine* (have an affine left inverse W_n^{-1})
- Mapping from the domain polygon to a 3D point:

$$S \circ W_n$$

Conversion as simplex composition

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$$S \circ W_n = S \circ W_n \circ (W_4^{-1} \circ W_4)$$

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- The 4-sided formulation is the composition of 3 simplexes:
  - W_4^{-1}: defined by the vertices of the rectangular domain
  - S: the S-patch (with homogenized control points)
  - W_n: ??? [a rational Bézier simplex of degree $n-2$]
- Composition:
  - Two algorithms (simple vs. efficient) [see the paper]

Determining the control points of W_n – homogenization

$$\lambda_i(p) = \frac{\prod_{j \neq i-1,i} D_j(p)}{\sum_{k=1}^{n} \prod_{j \neq k-1,k} D_j(p)}$$

- $D_j(p)$ is the signed distance of p from the j-th side
- Rational expression $\Rightarrow$ homogenized coordinates
  - Use the barycentric coordinates as "normal" coordinates
  - $(x, y, z) \equiv (wx, wy, wz, w(1 - x - y - z))$
- Homogenized form of W_n:

$$\left\{ \prod_{j \neq i-1,i} D_j(p) \right\}$$

Determining the control points of W_n – polarization

For any homogeneous polynomial $Q(u)$ of degree d, $\exists q$ s.t.

$$\begin{aligned} q(u_1, \dots, u_d) &= q(u_{\pi_1}, \dots, u_{\pi_d}), \\ q(u_1, \dots, \alpha u_{k_1} + \beta u_{k_2}, \dots, u_d) &= \alpha q(u_1, \dots, u_{k_1}, \dots, u_d) \\ &\quad + \beta q(u_1, \dots, u_{k_2}, \dots, u_d), \\ q(u, \dots, u) &= Q(u). \end{aligned}$$

Then q is called the *blossom* of Q.
The control points of its Bézier simplex form are

$$P^Q_{\mathbf{s}} = q(\underbrace{V_1, \dots, V_1}_{s_1}, \underbrace{V_2, \dots, V_2}_{s_2}, \dots, \underbrace{V_n, \dots, V_n}_{s_n}),$$

where V_i are the vertices of the simplex.

Determining the control points of W_n – blossom

- The blossom of W_n is

$$q(p_1, \dots, p_{n-2})_i = \frac{1}{(n-2)!} \cdot \sum_{\pi \in \Pi(n-2)} \prod_{\substack{k=1 \\ j \neq i-1, i}}^{n-2} D_j(p_{\pi_k})$$

  - $\Pi(n-2)$ is the set of permutations of $\{1, \dots, n-2\}$
  - k runs from 1 to $n-2$ while j from 1 to n skipping $i-1$ and i
- With this, the control points can be computed
- Simplex composition gives the quadrilateral S-patch
- Convert to "normal" homogeneous coordinates (wx, wy, wz, w)

Explicit formula for tensor product control points

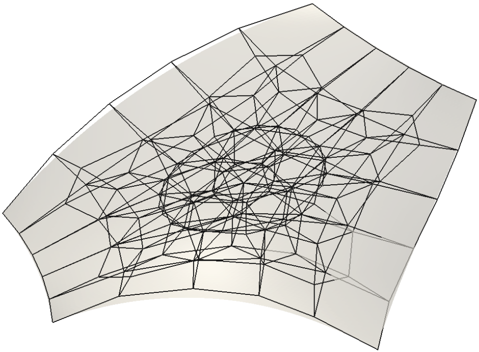
An n-sided S-patch of depth d can be represented as

$$\hat{S}(u,v) = \sum_{i=0}^{d} \sum_{j=0}^{d} C_{ij} B_i^d(u) B_j^d(v),$$

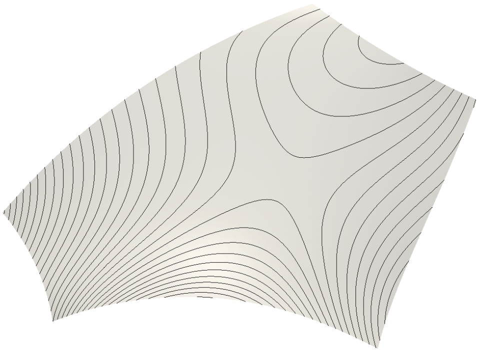
where

$$C_{ij} = \sum_{\substack{\mathbf{s} \\ s_2+s_3=i \\ s_3+s_4=j}} \frac{\binom{d}{\mathbf{s}}}{\binom{d}{i}\binom{d}{j}} P_{\mathbf{s}}.$$

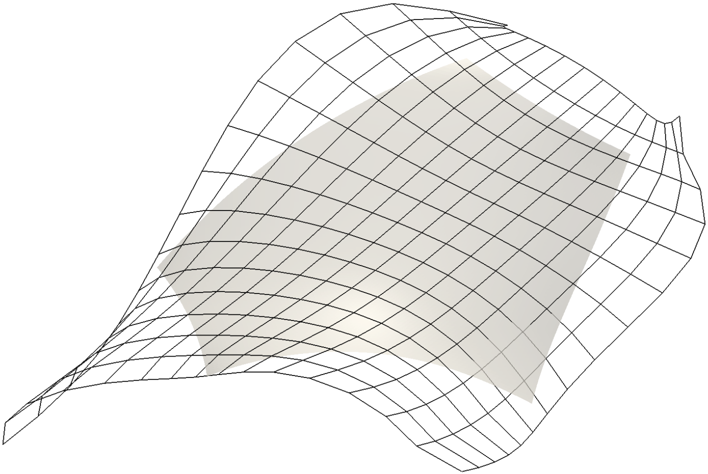
Converting a 5-sided patch – control net



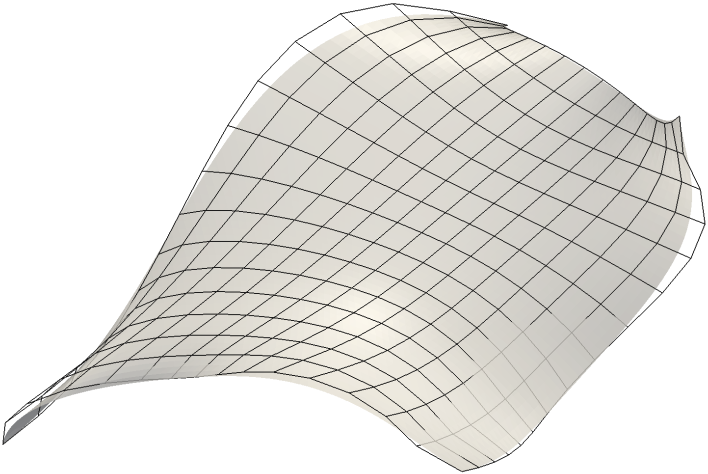
Converting a 5-sided patch – contours



Converting a 5-sided patch – trimmed tensor product



Converting a 5-sided patch – untrimmed tensor product



Limitations

- Efficiency
  - $n = 5$, $d = 8$ took > 5 minutes on a modern machine
    (How long would it have taken in 1989?)
  - Much faster algorithm is developed (see our upcoming paper)
- 3-sided patches
  - For Bézier triangles, the resulting patch is not rational
  - But there are simple alternative methods, e.g. [1992, Warren]
- Control net quality
  - Singularities on a circle around the domain
    - Denominator of Wachspress coordinates vanishes
  - Unstable control points near the corners
- Conclusion
  - The algorithm works, but it is not practical

Any questions?

Thank you for your attention.