Weingarten maps

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Let $\mathcal{S}$ denote an operator on a surface such that $\mathcal{S}(\mathbf{w})$ is the (negated) derivative of the unit normal by $\mathbf{w}$, i.e., for a fixed point $\mathbf{p}$ of the surface $\mathbf{x}(u, v)$,

$$\mathcal{S}(\mathbf{w}) = -\nabla_{\mathbf{w}} \mathcal{G},$$

where $\mathcal{G}$ is the Gauss map of the surface, and $\mathbf{w}$ is a vector in the tangent plane. This is the *shape operator*, or Weingarten map—a symmetric, linear operator, expressible by a 2×2 matrix (which is also symmetric when the basis vectors are perpendicular). Note that $\mathcal{S}$ is independent of the parameterization of $\mathbf{x}$ as long as the basis it is expressed in is parameterization-independent.

This matrix has very nice properties:

$$\begin{aligned} k(\mathbf{w}) &= \langle \mathcal{S}(\mathbf{w}), \mathbf{w} \rangle && \text{(normal curvature)} \\ K &= |\mathcal{S}|, && \text{(Gaussian curvature)} \\ H &= \mathrm{tr}(\mathcal{S})/2, && \text{(mean curvature)} \end{aligned}$$

and also the principal curvatures κ_1, κ_2 are the eigenvalues of $\mathcal{S}$, and the principal directions $\mathbf{e}_1$, $\mathbf{e}_2$ are the corresponding eigenvectors (expressed in the basis of the matrix).

If we write the matrix of $\mathcal{S}$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, then the above properties become obvious:

$$\mathcal{S} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \tag{1}$$

But this offers little help when we do not *know* these values.

Luckily $\mathcal{S}$ is easily expressible in the basis of the derivatives of $\mathbf{x}$:

$$\mathcal{S} = \mathbf{I}^{-1}\mathbf{I\!I}, \tag{2}$$

where $\mathbf{I}$ and $\mathbf{I\!I}$ are the first and second fundamental forms:

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \mathbf{I\!I} = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle, & F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle, & G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \\ L &= \langle \mathbf{n}, \mathbf{x}_{uu} \rangle, & M &= \langle \mathbf{n}, \mathbf{x}_{uv} \rangle, & N &= \langle \mathbf{n}, \mathbf{x}_{vv} \rangle, \end{aligned}$$

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

The drawback is that now the matrix of $\mathcal{S}$ is not independent of the parameterization.

Take Eq. (1) and add the unit normal $\mathbf{n}$ as a third basis vector, i.e., using the basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{n}$), bringing it into 3D space:

$$\mathcal{W} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is the *embedded* Weingarten map, sometimes also called the curvature tensor (but this term is abused). All the properties still stand, except for the Gaussian curvature, which can be expressed as

$$K = \frac{\mathrm{tr}(\mathcal{W})^2 - \mathrm{tr}(\mathcal{W}^2)}{2},$$

and that there is an extra 0 eigenvalue; normal curvatures are computed based on the projection of the given vector into the tangent plane.

Now we can also use the axes of 3D space as a basis—which is also independent of parameterization—and write the matrix as

$$\mathcal{W} = (\mathbf{J}^+)^\mathsf{T} \cdot \mathbf{I\!I} \cdot \mathbf{J}^+, \tag{3}$$

where $\mathbf{J}^+$ is the left pseudoinverse of the Jacobian:

$$\mathbf{J}^+ = (\mathbf{J}^\mathsf{T}\mathbf{J})^{-1}\mathbf{J}^\mathsf{T} = \mathbf{I}^{-1}\mathbf{J}^\mathsf{T} = (\mathbf{J} \cdot \mathbf{I}^{-1})^\mathsf{T}.$$

This formulation also has the advantage that its eigenvectors give the principal directions directly expressed in 3D coordinates. Note that while this matrix is exactly the same, independently of the parameterization, in exchange it *does* depend on the coordinates, so e.g. a rotation in 3D changes the elements of the matrix.

Implicit surfaces

For an implicitly defined surface $f(x, y, z) = 0$ take two arbitrary perpendicular vectors ($\mathbf{u}$,$\mathbf{v}$) in the tangent plane, and let

$$f_{uu} = \mathbf{u}^\mathsf{T}\mathbf{H}\,\mathbf{u}, \qquad f_{uv} = \mathbf{u}^\mathsf{T}\mathbf{H}\,\mathbf{v}, \qquad f_{vv} = \mathbf{v}^\mathsf{T}\mathbf{H}\,\mathbf{v},$$

where $\mathbf{H}$ is the Hessian of f. If we denote the norm of the gradient with $f_n = \|\nabla f\|$, the shape operator can be expressed in ($\mathbf{u}$,$\mathbf{v}$) as

$$\mathcal{S} = \frac{1}{f_n} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}. \tag{4}$$

Solving the characteristic equation $(\mathcal{S} - \mathbf{I}_2\kappa_i)\mathbf{e}_i = 0$ with $i \in \{1, 2\}$ gives

$$\mathbf{e}_i = \mathbf{u}f_{uv} + \mathbf{v}(\kappa_i f_n - f_{uu}) \quad \text{or} \quad \mathbf{e}_i = \mathbf{v}f_{uv} + \mathbf{u}(\kappa_i f_n - f_{vv}),$$ [1]

so we can compute the principal directions without eigendecomposition, since κ_i can be derived from

$$\kappa_i = H \pm \sqrt{H^2 - K}.$$

[1] Care should be taken to choose the formula that does not result in a null vector.

The embedded Weingarten map can be computed as

$$\mathcal{W} = \frac{1}{\|\nabla f\|} \cdot \mathbf{T} \cdot \mathbf{H} \cdot \mathbf{T}, \tag{5}$$

where $\mathbf{T} = \mathbf{I}_3 - \mathbf{n}\mathbf{n}^\mathsf{T}$ is the orthogonal projector onto the tangent plane, with $\mathbf{n} = \nabla f / \|\nabla f\|$ being the unit normal to the surface. Note that when $\|\nabla f\| = 1$ (i.e., when f is a signed distance function), the embedded Weingarten map is equal to the Hessian.

The formulas for the Gaussian and mean curvatures can be expressed succintly as below:

$$K = \frac{\nabla f^\mathsf{T} \cdot \mathrm{adj}(\mathbf{H}) \cdot \nabla f}{\|\nabla f\|^4}, \qquad H = \frac{\|\nabla f\|^2 \cdot \mathrm{tr}(\mathbf{H}) - \nabla f^\mathsf{T} \cdot \mathbf{H} \cdot \nabla f}{2\|\nabla f\|^3}.$$

Finally, a note on the sign of curvature values. They are relative to the orientation of the normal vector; conventionally they are defined such that in the implicit case (Eqs. 4 and 5) they are positive when the surface curves *away* from the normal direction, while in the parametric case (Eqs. 2 and 3) the resulting curvatures have the opposite sign.

References

On the shape operator, see any decent differential geometry textbook, e.g.:

- Barrett O'Neill, 2006: *Elementary Differential Geometry*
  (Elsevier, Revised 2nd Edition).
- Andrew Pressley, 2010: *Elementary Differential Geometry*
  (Springer, 2nd Edition).
- Wolfgang Kühnel, 2015: *Differential Geometry: Curves—Surfaces—Manifolds*
  (AMS, 3rd Edition, trans. by Bruce Hunt).

On the embedded Weingarten map, see

- Jörg Peters & Ulrich Reif, 2008: *Subdivision Surfaces*
  (Springer).

Finally, on the curvatures of implicit surfaces, see the following papers:

- Ron Goldman, 2005: *Curvature formulas for implicit curves and surfaces*
  (Computer Aided Geometric Design 22(7), pp. 632–658).
- Nicole Lehmann & Ulrich Reif, 2012: *Notes on the curvature tensor*
  (Graphical Models 74(6), pp. 321–325).
- Eric Albin &al., 2017: *Computational assessment of curvatures and principal directions of implicit surfaces from 3D scalar data*
  (Mathematical Methods for Curves and Surfaces [LNCS 10521], pp. 1–22).