Log-aesthetic curves and generalized Archimedean spirals

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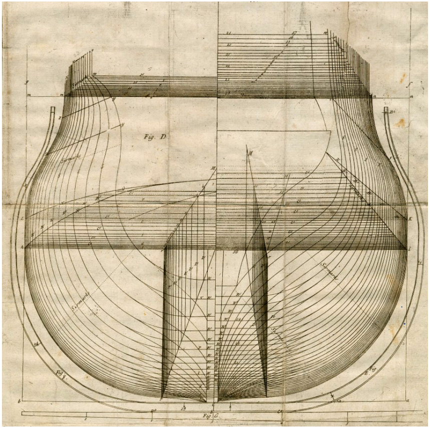
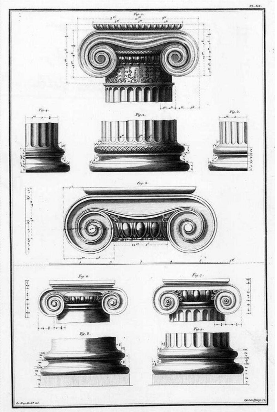
Classical Aesthetic Curves

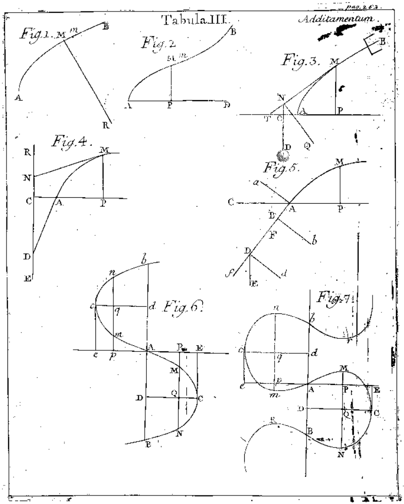


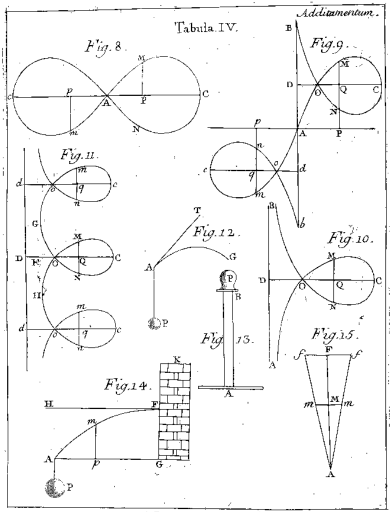


Spline energies [Hoschek–Lasser '96]

$$\kappa''(s) = 0 \qquad \text{(wooden)} \qquad \Rightarrow \text{clothoid}$$

$$\int \kappa(s)^2 \, \mathrm{d}s \to \min \qquad \text{(mechanical)} \qquad \Rightarrow \text{elastica}$$





Logarithmic Curvature Histogram (LCH)

Curve shape evaluation [Harada et al. '99]:

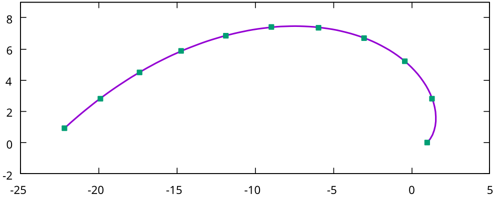
1. Take samples of the curvature radius (ρ_i) at equal arc lengths
2. Divide $\ln(\rho_i)$ into a fixed number of bins
3. Plot the logarithm of the percentage of samples in the bins

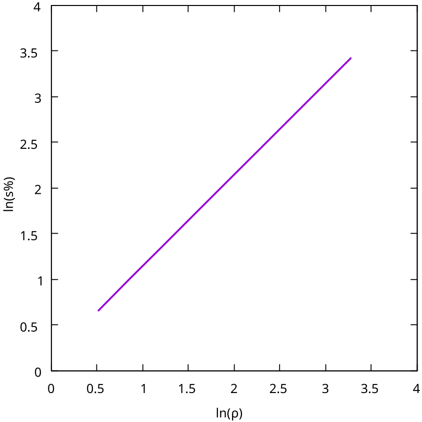
$\rightarrow$: $\ln \rho$

$$\uparrow : \ln \frac{\partial s}{\partial \ln \rho} = \ln \frac{\partial s}{\partial \rho / \rho}$$

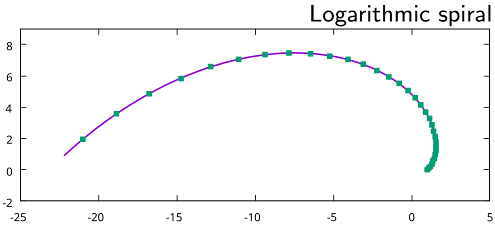
Straight lines are favorable

Logarithmic spiral



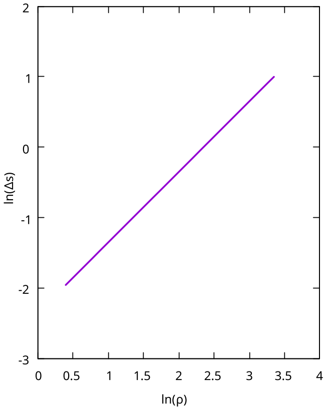


LCH—Alternative Interpretation



[Yoshida–Saito '06]:

1. Divide the curve into segments with the same $\Delta\rho/\rho$ ratio
2. Draw the log–log plot of segment lengths, i.e., $\ln(\Delta s)$ over $\ln(\rho)$



Linearity means

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

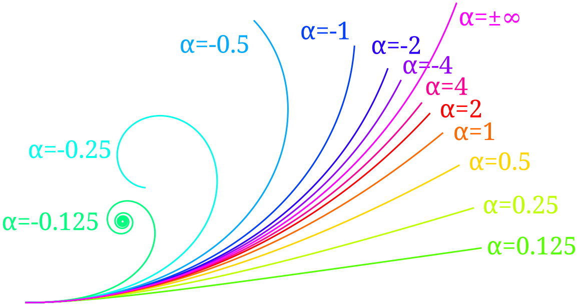
where α is the slope

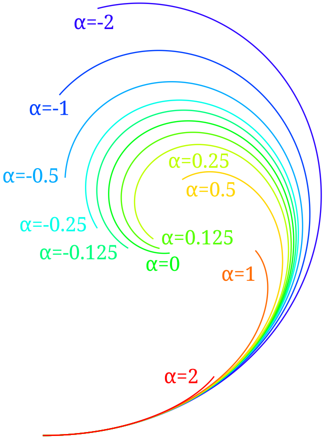
Log-Aesthetic Curves [Miura '06]

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

$$\theta(s) = \frac{\alpha(c_0 s + c_1)^{(\alpha-1)/\alpha}}{(\alpha - 1)c_0} + c_2$$

$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s\right)$$





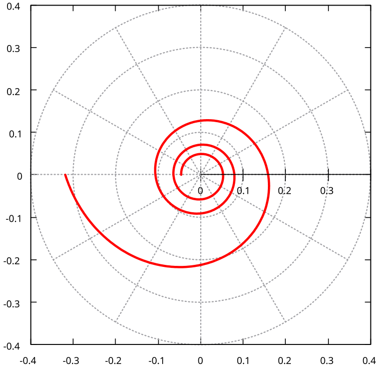
Types of Log-Aesthetic Curves

- Circle ($c_0 = 0$)
- Circle involute ($\alpha = 2$)
- Logarithmic spiral ($\alpha = 1$)
  - $\theta(s) = \ln(c_0 s + c_1)/c_0 + c_2$
- Nielsen's spiral ($\alpha = 0$)
  - $\kappa(s) = \exp(c_0 s + c_1)$
  - $\theta(s) = \exp(c_0 s + c_1)/c_0 + c_2$
- Clothoid ($\alpha = -1$)

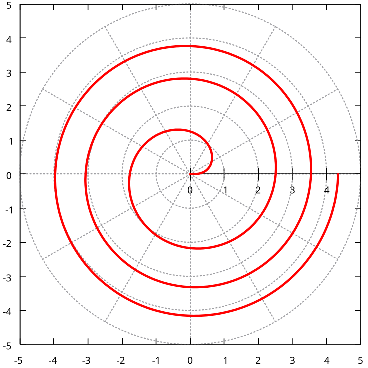
Generalized Archimedean Spirals

Polar equation: $r = a + b\phi^{\frac{1}{c}}$

- $c = -2$: lituus
- $c = -1$: hyperbolic spiral
- $c = 1$: Archimedean (arithmetic) spiral
- $c = 2$: Fermat's spiral



Hyperbolic spiral



Fermat's spiral

Radial Curves

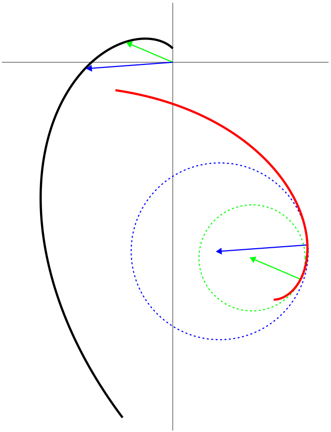
- Vector to the center of curvature, placed at the origin
- $\theta(t)$: tangent angle to the x axis
- $\theta^{\perp}(t) = \theta(t) + \frac{\pi}{2}$
- $\mathbf{R}(t) = [\cos\theta^{\perp}(t), \sin\theta^{\perp}(t)] \cdot \rho(t)$
- For log-aesthetic curves:

$$\rho(\theta^{\perp}) = \left(\theta^{\perp} c_0 \frac{\alpha - 1}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

- Polar equation:

$$r = b\phi^{\frac{1}{\alpha-1}}$$

- GA spiral with $a = 0$ and $c = \alpha - 1$

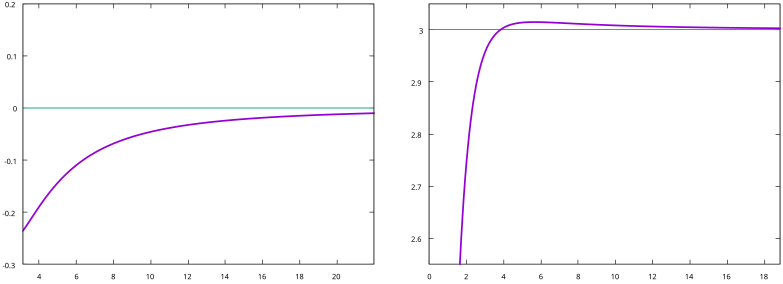


Logarithmic spiral
(special case: $r = e^{b\phi}$)

LCH Slope of GA spirals

$$\alpha(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t)\right)$$

Approaches $c + 1$ (slope of the related LA curve)



$c = -1$ (Hyperbolic spiral)

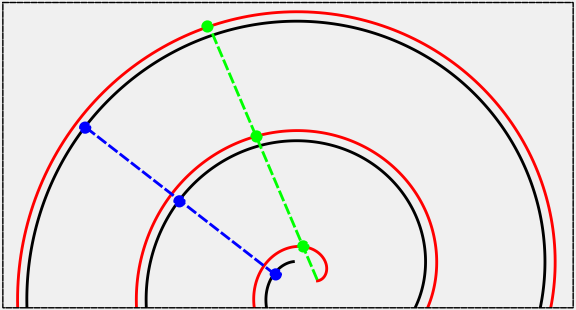
$\rightarrow \alpha = 0$ (Nielsen's spiral)

$c = 2$ (Fermat's spiral)

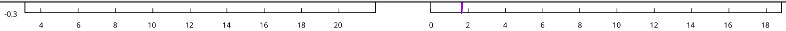
$\rightarrow \alpha = 3$

LCH Slope of GA spirals

$$\alpha(t) = 1 + \frac{\rho(t)}{\dots}\left(\frac{\rho'(t)s''(t)}{\dots} - \rho''(t)\right)$$



Archimedes' spiral & Circle involute



$c = -1$ (Hyperbolic spiral)

$\to \alpha = 0$ (Nielsen's spiral)

$c = 2$ (Fermat's spiral)

$\to \alpha = 3$

Approximating LA curves by GA spirals

- LA curve segment:

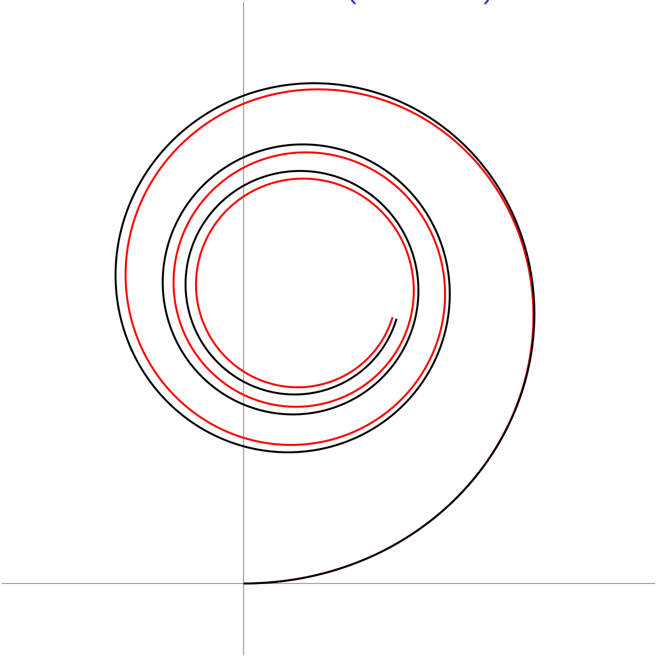
$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s \right), \quad s \in [s_{\min}, s_{\max}]$$

- GA spiral segment:

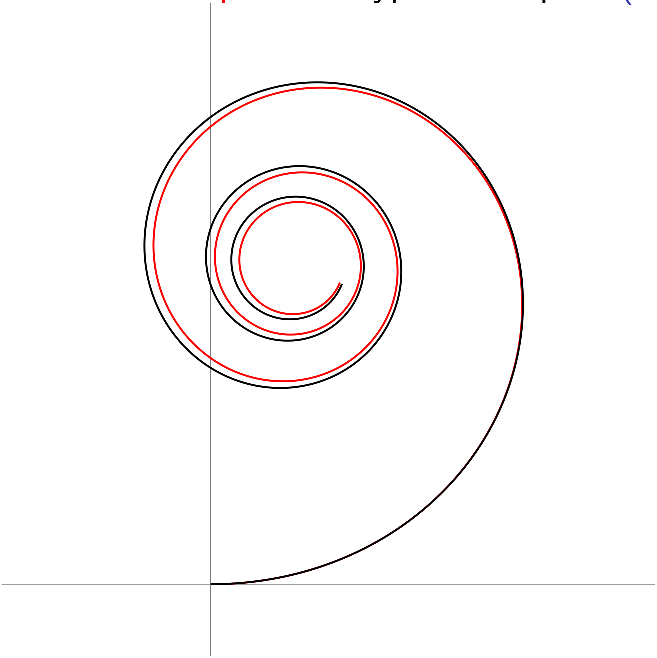
$$\mathbf{C}_{\mathrm{GA}}(t) = [\cos t, \sin t] \cdot (a + bt^{\frac{1}{c}}), \quad t \in [t_{\min}, t_{\max}]$$

- $a = 0$ and $c = \alpha - 1$
- Assume matching starting point and direction
  - Simple translation/rotation
- Interpolate curvature at $t_{\min}$
  - If $t_{\min}$ is known $\rightarrow$ b can be computed
- Interpolate curvature derivative at $t_{\min}$
  - $t_{\min}$ found by binary search
  - Initial frame by iterative doubling

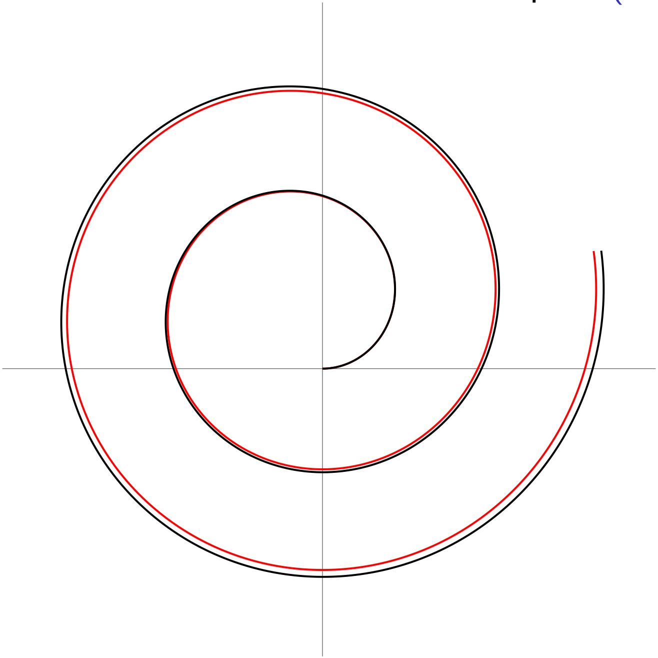
Example 1: clothoid vs. lituus ($\alpha = -1$)



Example 2: Nielsen's spiral vs. hyperbolic spiral ($\alpha = 0$)



Example 3: Circle involute vs. arithmetic spiral $(\alpha = 2)$



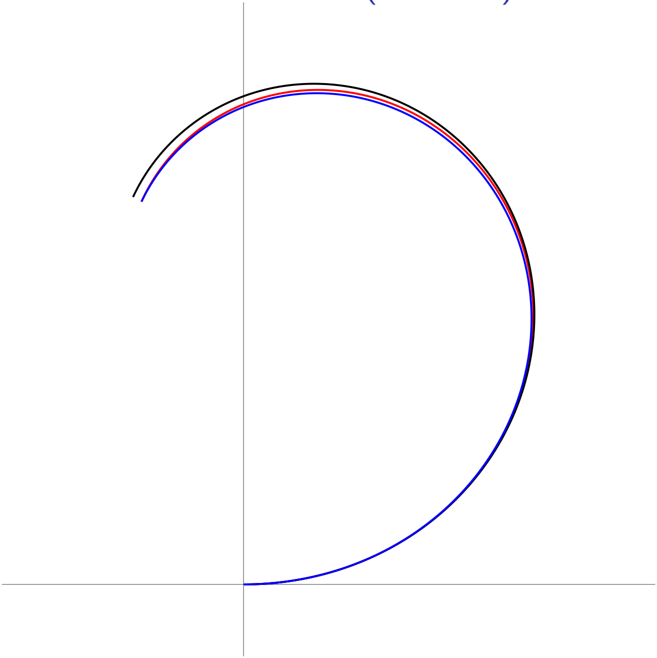
Alternative Constraint

- Idea: Fix the endpoint instead of the curvature derivative
- Different error function for the bisection search
  - Radial distance of the endpoint to the GA spiral

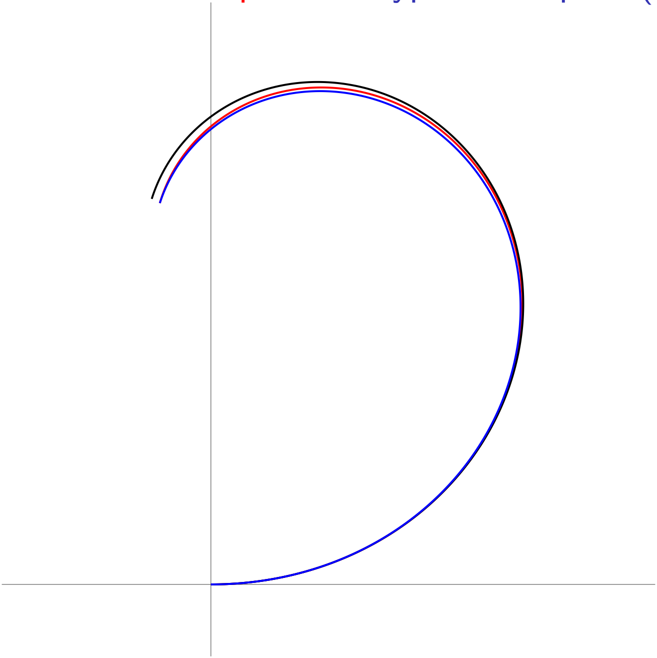
Algorithm

1. Rotate the spiral s.t. $\mathbf{C}'_{\mathrm{GA}}(t_{\mathsf{min}})$ points to $\theta(s_{\mathsf{min}})$.
2. Set $\mathbf{Q}$ (the spiral center) s.t. $\mathbf{Q} + \mathbf{C}_{\mathrm{GA}}(t_{\mathsf{min}}) = \mathbf{P}_0$.
3. Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors from $\mathbf{Q}$ to $\mathbf{P}_0$ and $\mathbf{C}(s_{\mathsf{max}})$.
4. Set $t_{\mathsf{max}} = t_{\mathsf{min}} + \arccos\langle \mathbf{u}, \mathbf{v} \rangle$, or, if $\det(\mathbf{u}, \mathbf{v}) < 0$, choose the larger angle: $t_{\mathsf{max}} = t_{\mathsf{min}} + 2\pi - \arccos\langle \mathbf{u}, \mathbf{v} \rangle$.
5. The error is $\|\mathbf{C}(s_{\mathsf{max}}) - \mathbf{Q}\| - \|\mathbf{C}_{\mathrm{GA}}(t_{\mathsf{max}})\|$.

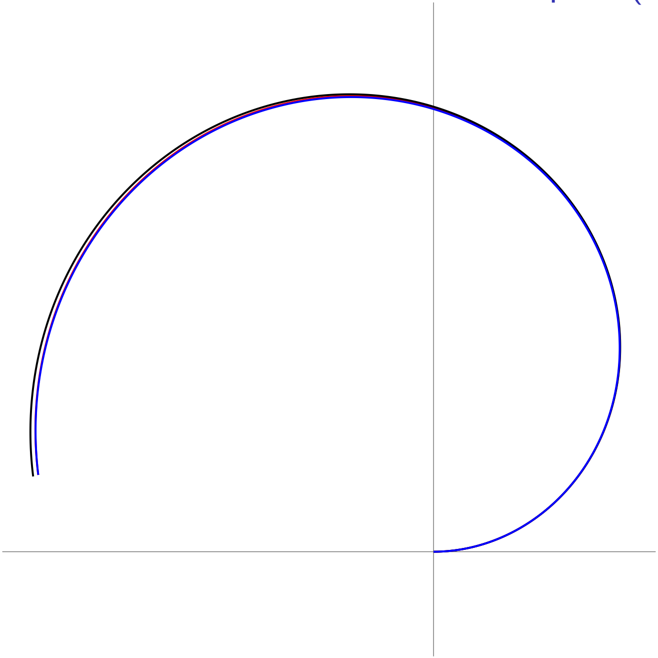
Example 1b: clothoid vs. lituus ($\alpha = -1$)



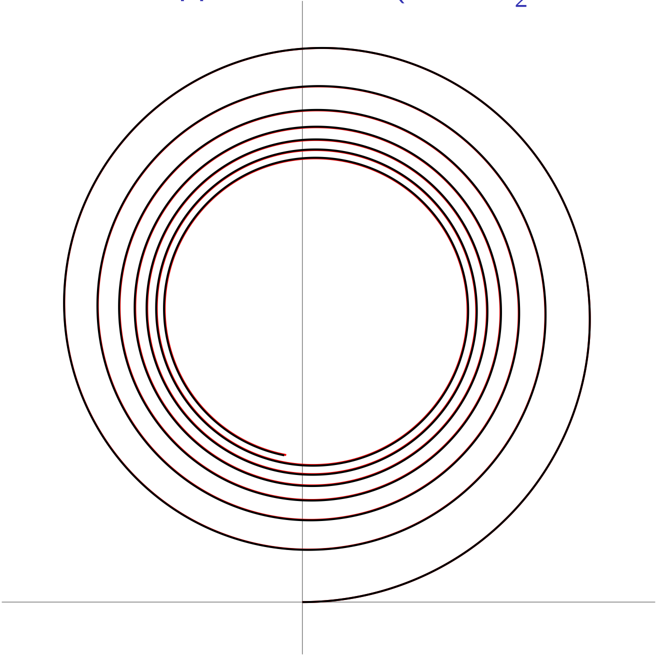
Example 2b: Nielsen's spiral vs. hyperbolic spiral ($\alpha = 0$)



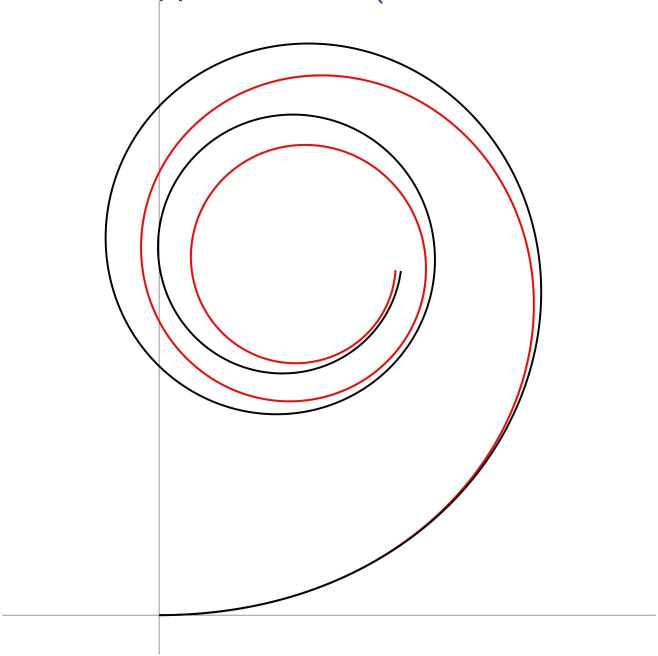
Example 3b: Circle involute vs. arithmetic spiral ($\alpha = 2$)



Example 4: Good Approximation ($\alpha = -\frac{3}{2}$, $t_{\min} \approx 9.88$)



Example 5: Bad Approximation ($\alpha = -1$, $t_{\min} \approx 1.42$)



How Good is the Approximation?

- Works well for shorter segments
- Analyze the $c_0 = c_1 = 1$, $c_2 = 0$ case
- Arcs get more circular as t_{min} (θ_0) grows
- Maximum radial rotation ($t_{\mathrm{max}} - t_{\mathrm{min}}$) under tolerance:

$\alpha \rightarrow$	-2	$-3/2$	-1	$-1/2$	0	3/2	2
$\theta_0 = 0$	1.82	1.61	1.38	1.13	2.50	1.35	1.48
$\theta_0 = \pi/4$	2.61	2.35	2.03	1.69	2.99	1.94	2.62
$\theta_0 = \pi/2$	3.31	3.06	2.73	2.29	3.43	2.52	3.82
$\theta_0 = 3\pi/4$	4.09	3.73	3.33	2.89	3.89	3.13	4.79
$\theta_0 = \pi$	4.78	4.44	3.99	3.42	4.37	3.77	$> 2\pi$
$\theta_0 = 5\pi/4$	5.40	5.04	4.61	4.00	4.79	4.34	$> 2\pi$
$\theta_0 = 3\pi/2$	6.03	5.62	5.15	4.56	5.19	4.80	$> 2\pi$
$\theta_0 = 7\pi/4$	$> 2\pi$	6.22	5.68	5.05	5.58	5.59	$> 2\pi$
$\theta_0 = 2\pi$	$> 2\pi$	$> 2\pi$	6.24	5.51	5.97	6.25	$> 2\pi$

[tolerance: 10^{-3} of the bounding rectangle diagonal]

Reconstructing Log-Aesthetic Curves from Radials

- From the construction: $\|\mathbf{C}'(t)\| = \|\mathbf{R}(t)\|$
- Inverse radial:

$$\mathbf{C}(t) = \int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \mathbf{R}(t)\,\mathrm{d}t$$

- Explicit equations for some cases, e.g.:
  - $b = 1$, $c = 1$ (circle involute):

$$[t\cos t - \sin t, t\sin t + \cos t]$$

  - $b = 1$, $c = \frac{1}{2}$:

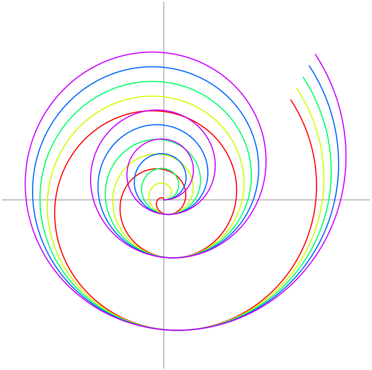
$$[(t^2 - 2)\cos t - 2t\sin t, (t^2 - 2)\sin t + 2t\cos t]$$

  - etc.
- May involve incomplete gamma functions

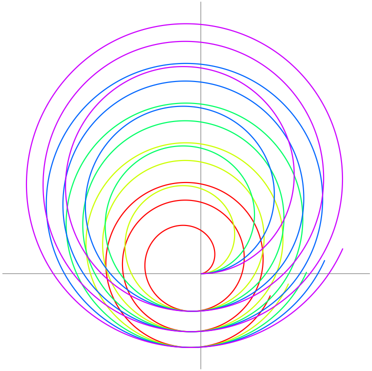
Generalized Log-Aesthetic Curves

What if $a \neq 0$?

- Arithmetic spirals ($c = 1$): just a shift
- $c < 0$: LCH slope diverges into $\pm\infty$
- $c > 0$: Still converges to $c + 1$



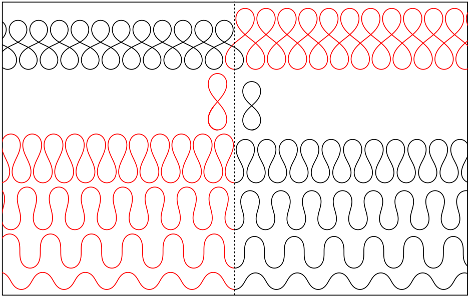
$c = \frac{1}{2}$, $a \in \{0, 20, 40, 60, 80\}$

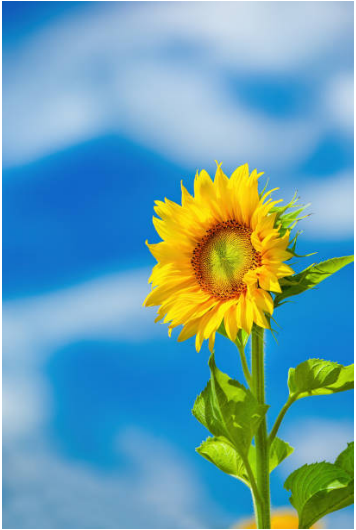


$c = 2$, $a \in \{0, 1, 2, 3, 4\}$

Conclusion

- Radial of LA curve $\rightarrow$ GA spiral
- LCH slope of radials approach α
- GA spirals approximate LA curves
- Reconstruction from radials
- Generalized LA curves





← Elastica vs. trig-aesthetic curves

[Salvi '26]

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