A note on invariants of aesthetic curve families

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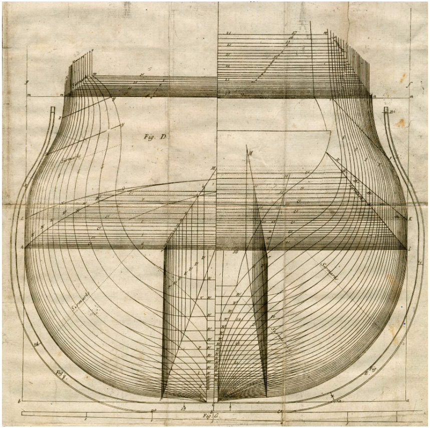
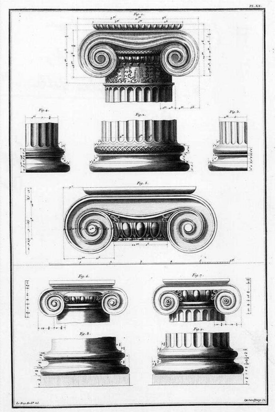
Outline

- What Are Aesthetic Curves?
  - Classical Aesthetic Curves
  - Logarithmic Curvature Histogram
  - Log-Aesthetic Curves
  - Trig-Aesthetic Curves
  - Elastica
- Invariants of Curve Families
  - LAC–TAC Common Form
- Conclusion

MŰEGYETEM 1782

Classical Aesthetic Curves

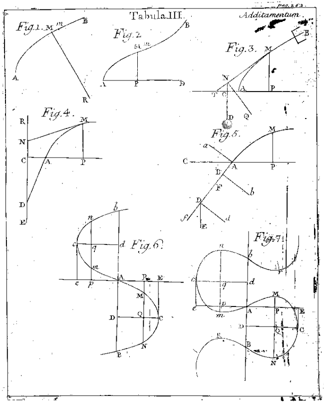


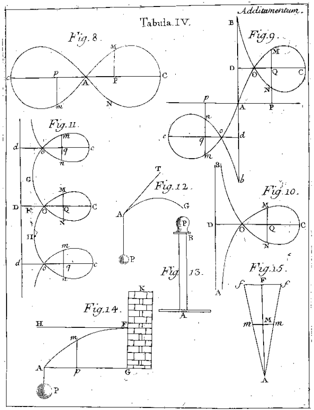


Spline energies

$$\kappa''(s) = 0 \qquad \text{(wooden)} \qquad \Rightarrow \text{clothoid}$$

$$\int \kappa(s)^2 \,\mathrm{d}s \to \min \qquad \text{(mechanical)} \qquad \Rightarrow \text{elastica}$$





L. Euler: *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*. Lausanne & Geneve, 1744.

J. Hoschek, D. Lasser: *Fundamentals of Computer Aided Geometric Design*. A. K. Peters, Wellesley, 1996.

Logarithmic Curvature Histogram (LCH)

Curve shape evaluation:

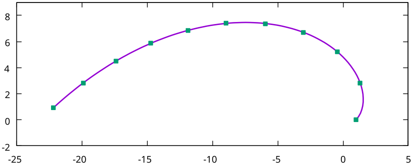
1. Take samples of the curvature radius (ρ_i) at equal arc lengths
2. Divide $\ln(\rho_i)$ into a fixed number of bins
3. Plot the logarithm of the percentage of samples in the bins

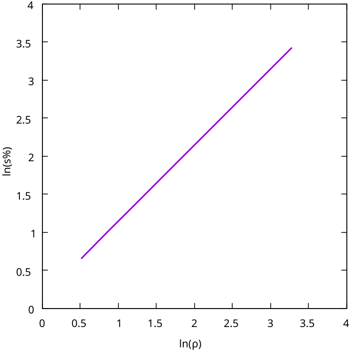
$$\rightarrow : \ln \rho$$

$$\uparrow : \ln \frac{\partial s}{\partial \ln \rho} = \ln \frac{\partial s}{\partial \rho / \rho}$$

Straight lines are favorable

Logarithmic spiral





T. Harada, F. Yoshimoto, M. Moriyama: *An aesthetic curve in the field of industrial design*. Proceedings of IEEE Symposium on Visual Language, pp. 38–47, 1999.

LCH—Alternative Interpretation

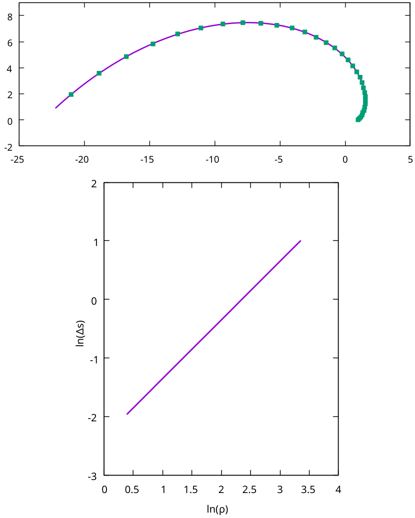
1. Divide the curve into segments with the same $\Delta\rho/\rho$ ratio
2. Draw the log–log plot of segment lengths, i.e., $\ln(\Delta s)$ over $\ln(\rho)$

Linearity means

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

where α is the slope

Logarithmic spiral



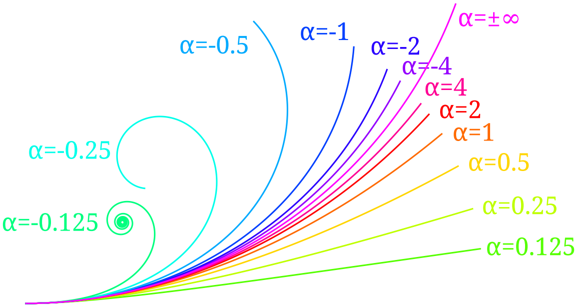
N. Yoshida, T. Saito: *Interactive aesthetic curve segments*. TVC 22(9-11):896–905, 2006.

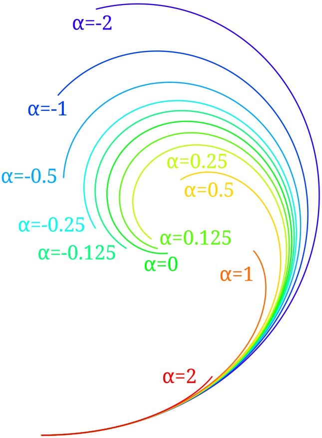
Log-Aesthetic Curves

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha}$$

$$\theta(s) = \frac{\alpha(c_0 s + c_1)^{(\alpha-1)/\alpha}}{(\alpha - 1)c_0} + c_2$$

$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s\right)$$





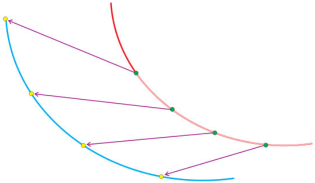
K. T. Miura: *A general equation of aesthetic curves and its self-affinity*. CAD&A(1–4):457–464, 2006.

Types of Log-Aesthetic Curves

- Circle ($c_0 = 0$)
- Circle involute ($\alpha = 2$)
- Logarithmic spiral ($\alpha = 1$)
  - $\theta(s) = \ln(c_0 s + c_1)/c_0 + c_2$
- Nielsen's spiral ($\alpha = 0$)
  - $\kappa(s) = \exp(c_0 s + c_1)$
  - $\theta(s) = \exp(c_0 s + c_1)/c_0 + c_2$
- Clothoid ($\alpha = -1$)

Properties

- Self-affinity
  - Weaker than self-similarity
  - The "tail" of a log-aesthetic arc can be affinely transformed into the whole curve
- Natural shape
  - Egg contour, butterfly wings, etc.
- Also appears in art and design
  - Japanese swords, car bodies, etc.
- Close connection to (generalized) Archimedean spirals



Scaling a segment shows self-affinity

A swallowtail butterfly

A japanese sword

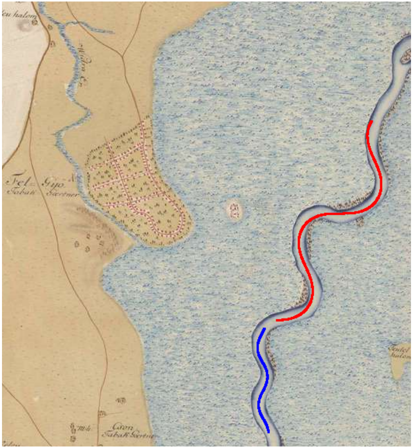
P. Salvi: *Log-aesthetic curves and generalized Archimedean spirals*. CAGD 121:102468, 2025.

Trig-Aesthetic Curves

$$\kappa(s) = c_0 \cos(c_1 s + c_2), \quad \theta(s) = \frac{c_0}{c_1} \sin(c_1 s + c_2) + c_3$$

- ‘Sine-generated curves’
- Used in geophysics (models river meandering)
- c_0: scaling
- c_1: **shape**
- c_2: starting parameter
- c_3: starting tangent
- Simpler version:

$$\kappa(s) = \cos(s/c)$$
$$\theta(s) = c \sin(s/c)$$



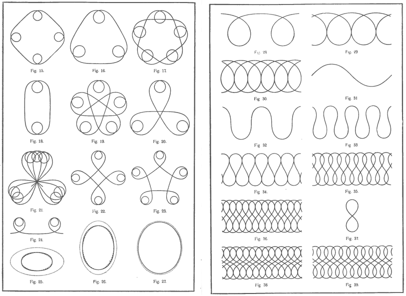
Meanders of the Tisza river (XIX. c.)

P. Salvi: *Generalized catenaries and trig-aesthetic curves.* CAD&A 23(1):56–67, 2026.

Eduard Lehr (1906–1955)

- Doctoral thesis (1932) on

$$\kappa(s) = a + b\cos(cs)$$




6.6.47.
Dr. Eduard Lehr

- Precise drawings using...
  - Planimeter
  - Mechanical calculator
  - Slide rule

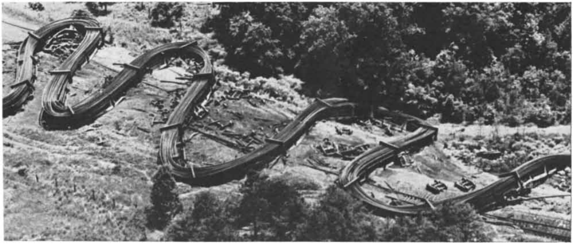
- Connection to Elastica

P. Salvi: *The curves of Eduard Lehr*. G - Slovak Journal for Geometry and Graphics 22(43):5–23, 2025.

Connection with Elastica

$$\kappa(s) = \cos(s/c), \quad \theta(s) = c \sin(s/c)$$

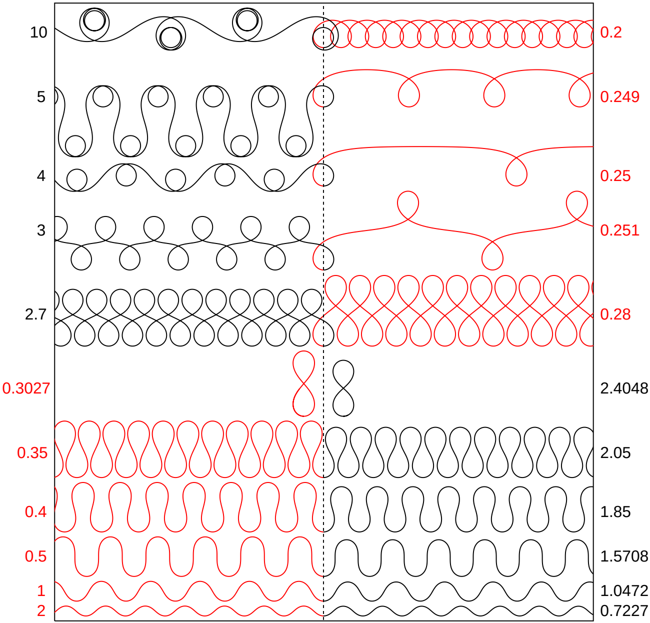
- Rivers meander along elastic curves
    - Most probable path of a particle turning by normal distribution
    - Minimize bending energy with fixed arc length
    - Solutions of $\theta''(s) + \lambda \sin \theta(s) = 0$
    - Maximum turning angle: $\arccos(1 - \frac{1}{2\lambda})$
- Trig-aesthetic curves are similar
    - Maximum turning angle: c



Wreck of a Southern Railway freight train near Greenville, S.C., 1965.

H. von Schelling: *Most frequent particle paths in a plane*. Eos 32(2):222–226, 1951.
W. B. Langbein, L. B. Leopold: *River meanders—Theory of minimum variance*. Technical Report 422-H, United States Geological Survey, 1966.

Trig-Aesthetic Curves vs. Elastica



Cesàro's Invariants

- Series of radii of curvature:

$$\rho_{(0)} = \rho, \qquad \rho_{(k)} = \rho\rho'_{(k-1)}$$

- $\rho_{(k)} = \frac{\mathrm{d}\rho_{(k-1)}}{\mathrm{d}\theta}$, all radii are of the same scale
- Invariant: $f(\rho, \rho_{(1)}, \ldots, \rho_{(k)}) \equiv 0$
- Example: parabola with stretch a (e.g. $y = ax^2 + bx + c$)
  - $\rho = \frac{\sec^3\theta}{2a} \quad \Rightarrow \quad 9\rho^2 + 4\rho_{(1)}^2 - 3\rho\rho_{(2)} \equiv 0$
- Proposition: use $f(\rho, \rho_{(1)}, \ldots, \rho_{(k)}) \equiv \text{const.}$
  - More information (eliminate only non-shape parameters)
  - Often more concise expressions
  - E.g. parabola: $(\rho_{(1)}^2/\rho^2 + 9)^3/\rho^2 \equiv (54a)^2$
    [not very concise and a is not a shape parameter]
- Better for aesthetic curves: κ, κ', κ'', ...
- ODE form: $\theta'' = f(\theta, \theta') \Rightarrow \kappa' = f(\theta, \kappa)$, useful for plotting

E. Cesàro: *Lezioni di geometria intrinseca*. Naples, 1896.

Table of Invariants

	Elastica
Intrinsic	$\mathrm{cn}(\sqrt{\lambda}s, \frac{1}{4\lambda})$
ODE	$-\lambda \sin\theta$
Constant	$\kappa'^2 + \kappa''^2/\kappa^2 = \lambda^2$
Invariant	$\kappa\kappa''' + \kappa'(\kappa^3 - \kappa'')$

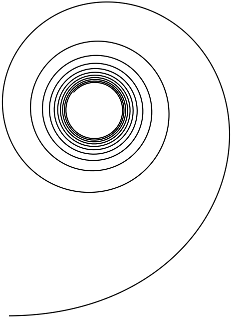
	Log-Aesthetic Curves ($\alpha \neq 0$)	Nielsen's spiral ($\alpha = 0$)
Intrinsic	$(s+1)^{-\frac{1}{\alpha}}$	$\exp(s)$
ODE	$-\kappa^{\alpha+1}/\alpha$	κ
Constant	$\kappa\kappa''/\kappa'^2 = \alpha + 1$	N/A
Invariant	$\kappa'^2\kappa'' + \kappa\kappa'\kappa''' - 2\kappa\kappa''^2$	$\kappa - \kappa'$

	Trig-Aesthetic Curves	Complex TAC
Intrinsic	$\cos(s/c)$	$\cosh(s/c)$
ODE	$-\theta/c^2$	θ/c^2
Constant	$(1-\kappa^2)/\kappa'^2 = c^2$	$\Leftarrow$
Invariant	$\kappa\kappa'^2 + \kappa''(1-\kappa^2)$	$\Leftarrow$

Circle / Clothoid / Nielsen's spiral / TAC common constant form: κ''/κ
LAC–TAC common constant form: $\kappa\kappa'''/\kappa'\kappa'' = 2\alpha + 1$ (Nielsen $\approx$ TAC)

LAC–TAC Common Form

- $(\kappa\kappa''')/(\kappa'\kappa'') = 2\alpha + 1$
- $(\ln\kappa'')'/(\ln\kappa)' = \hat{C}$
- $(\ln\kappa'')' = \hat{C}(\ln\kappa)'$
- $\ln\kappa'' = \hat{C}\ln\kappa + \ln\hat{A}$
- $\kappa'' = \hat{A}\kappa^{\hat{C}}$
- $\kappa'^2 = \frac{2\hat{A}}{\hat{C}+1}\kappa^{\hat{C}+1} + \hat{B}$
- $\Rightarrow \kappa'^2 = A\kappa^B + C.$
  - $B = 1$: generalized catenaries
  - $C = 0$: log-aesthetic curve $[\alpha = (B-2)/2]$
  - $B = 2$, $A = -C$: trig-aesthetic $[c = 1/\sqrt{C}]$
  - Other cases. . . ?

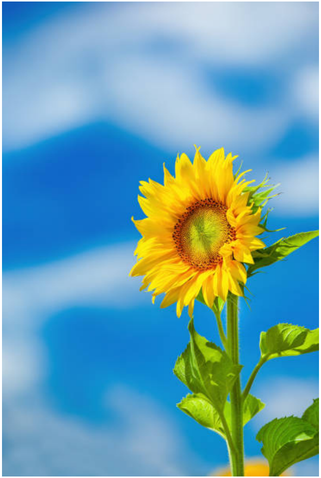


$$\kappa' = \sqrt{\frac{1}{\kappa} + 1}$$

$(A = 1,\ B = -1,\ C = 1)$

Conclusion

- Three curve families
    - Log-aesthetic curves
    - Trig-aesthetic curves
    - Elastica
- Invariants & constant forms
    - LAC–TAC common form



Spirals of sunflower seeds

`https://3dgeo.iit.bme.hu/`