

# A Note on Invariants of Aesthetic Curve Families

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## Abstract

We revisit the definition of invariants for curve families, show specific examples using aesthetic curves, and propose the use of a ‘constant form’ instead that also makes shape parameters directly accessible.

## 1. Introduction

Cesàro, in his seminal work on intrinsic equations<sup>1</sup> (Section II, Eq. 13), defines a series of radii of curvature as

$$\rho_{(0)} = \rho, \quad \rho_{(k)} = \rho \rho'_{(k-1)}, \quad (1)$$

where  $\rho = \frac{1}{\kappa}$  is the radius of curvature as a function of arc length ( $s$ ). An advantage of this definition (compared to simply using the derivatives) is that all such radii are of the same scale. It is easy to see that  $\rho_{(k)}$  is just the  $k$ th derivative of  $\rho$  w.r.t.  $\theta$ , the tangent angle, since

$$\frac{d\rho_{(k-1)}}{d\theta} = \rho'_{(k-1)} \cdot \frac{ds}{d\theta} = \rho'_{(k-1)} \cdot \frac{1}{\frac{d\theta}{ds}} = \rho'_{(k-1)} \cdot \frac{1}{\kappa} = \rho \rho'_{(k-1)}. \quad (2)$$

Then he goes on (in Section IV/8) to define the *invariant* of a curve family as an algebraic expression, involving the first  $k$  radii of curvature, that is constant zero. For example, the invariant of parabolas is

$$9\rho^2 + 4\rho_{(1)}^2 - 3\rho\rho_{(2)}, \quad (3)$$

as can be confirmed using the formula

$$\rho = \frac{\sec^3 \theta}{2a}, \quad (4)$$

where  $a$  is the stretch factor of the parabola (as in the equation  $y = ax^2 + bx + c$ ).

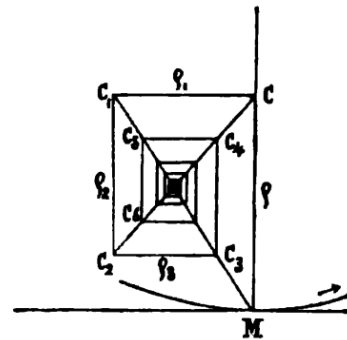
In the context of aesthetic curves, curvature and its derivatives with respect to arc length are much more common, so we adopt that notation for the rest of the paper. In the next section we show the invariant of various curve families, and argue that we get a better characteristic if we eliminate only

those parameters that do not play an active role in determining the shape of the curve (i.e., those associated with rigid motions, scaling, or a phase shift in the parameterization).

## 2. Invariants of aesthetic curve families

Starting from a differential equation form of a curve family, we can eliminate its parameters by successively taking derivatives. Let us take *log-aesthetic* curves<sup>3</sup> as our first example. Their conventional equation is

$$\kappa = (c_0 s + c_1)^{-\frac{1}{\alpha}}, \quad (5)$$



**Figure 1:** Cesàro's geometric representation of the sequence of curvature radii  $\rho, \rho_{(1)}, \rho_{(2)}, \dots$  through successive evolutes, shown in the special case of cycloidal curves, where the centers of curvature lie alternately on two lines rotating about a fixed point (Section II/13g), having the invariant  $\rho_{(1)}\rho_{(2)} - \rho\rho_{(3)}$  (Section IV/9a).

Family	Intrinsic equation	ODE form	Constant form	Invariant
Log-aesthetic ( $\alpha \neq 0$ )	$(s+1)^{-\frac{1}{\alpha}}$	$-\kappa^{\alpha+1}/\alpha$	$\kappa\kappa''/\kappa'^2$	$\kappa'^2\kappa''' + \kappa\kappa'\kappa'''' - 2\kappa\kappa''^2$
Nielsen's spiral	$\exp(s)$	$\kappa$	N/A	$\kappa - \kappa'$
Elastica	$\text{cn}(\sqrt{\lambda}s, \frac{1}{4\lambda})$	$-\lambda \sin \theta$	$\kappa'^2 + \kappa''^2/\kappa^2$	$\kappa\kappa'''' + \kappa'(\kappa^3 - \kappa'')$
Trig-aesthetic	$\cos(s/c)$	$-\theta/c^2$	$(1 - \kappa^2)/\kappa'^2$	$\kappa\kappa'^2 + \kappa''(1 - \kappa^2)$
Complex trig-aesthetic	$\cosh(s/c)$	$\theta/c^2$	—  —	—  —

**Table 1:** Equations, ODE/constant forms and invariants for different families.

but since we are not interested in scaled versions of the same curve, we can use the convention  $\kappa(0) = 1$ , thereby fixing  $c_1 = 1$ :

$$\kappa = (c_0s + 1)^{-\frac{1}{\alpha}}. \quad (6)$$

First we need to convert this to differential equation form, so we take the derivative:

$$\kappa' = -\frac{c_0}{\alpha}(c_0s + 1)^{-\frac{\alpha+1}{\alpha}}, \quad (7)$$

which yields

$$\kappa' = -\frac{c_0}{\alpha}\kappa^{\alpha+1}. \quad (8)$$

We will call the right-hand side of such equations as the *ODE form*. It is particularly useful for drawing these curves, as shown in Appendix A. Deriving further, we get

$$\kappa'' = -\frac{c_0}{\alpha}(\alpha+1)\kappa^\alpha\kappa' = (\alpha+1)\frac{\kappa'^2}{\kappa}, \quad (9)$$

which can also be written as

$$\frac{\kappa\kappa''}{\kappa'^2} = \alpha + 1. \quad (10)$$

With one final derivation step (retaining only the numerator), we obtain

$$\kappa'^2\kappa''' + \kappa\kappa'\kappa'''' - 2\kappa\kappa''^2, \quad (11)$$

which is the invariant for this family. Note, however, that Eq. (10), of the form  $f(\kappa, \kappa', \kappa'', \dots) = \text{const.}$ , is both more compact, and also provides more information. The  $c_0$  parameter is already removed (which is appropriate, since it only controls the scale and phase shift), but we still have access to the important shape parameter  $\alpha$ .

The  $\alpha = 0$  case (Nielsen's spiral) has the equation

$$\kappa = \exp(c_0s + c_1), \quad (12)$$

which, for the same reasons as above, can be simplified to

$$\kappa = \exp(s), \quad (13)$$

so the ODE form is trivially

$$\kappa' = \kappa. \quad (14)$$

Here we do not obtain any additional information, the invariant

$$\kappa - \kappa' \quad (15)$$

contains everything we need to know. Since this is really a special case of the log-aesthetic family, we could also have used (the slightly more complex) Eqs. (10) and (11). This shows that these invariants are hierarchical, and a more restrictive family has simpler expressions.

Similar computations can be done for other families, such as the *elastica* investigated by Bernoulli and Euler (see e.g. Levien<sup>2</sup>), with the constant being  $\lambda^2$ , or the recently proposed *trig-aesthetic* family<sup>5</sup>

$$\kappa(s) = \cos(s/c), \quad (16)$$

where the constant form gives  $c^2$  (see Table 1).<sup>†</sup>

As it was the case for Nielsen's spiral, we can see that the 'complex trig-aesthetic' family, which was proposed originally by using complex shape parameters in the trig-aesthetic equation, is compatible with the constant form and invariant of its "parent" family.

Another possible constant form for (both ordinary and complex) trig-aesthetic curves is  $\kappa/\kappa''$ , leading to the invariant

$$\kappa'\kappa'' - \kappa\kappa''', \quad (17)$$

but note that these, as opposed to those included in the table, are shared with  $\kappa = \exp(s/c)$ , i.e., Nielsen's spiral.

We can thus see how close log- and trig-aesthetic families are. Can we devise a formula that encompasses all of these? The answer is trivially *yes*, we just need to multiply the invariant forms; creating a common constant form is not so straightforward, but there *is* a simple solution:

$$\frac{\kappa\kappa'''}{\kappa'\kappa''} = 2\alpha + 1, \quad (18)$$

<sup>†</sup> The invariant of a larger family that includes trig-aesthetic curves was investigated by Eduard Lehr in the 1930s.<sup>4</sup>

which includes all trig-aesthetic curves in the same group ( $\alpha = 0$ ) as Nielsen's spiral.

Going the other way around, can we *separate* trig-aesthetic curves from Nielsen's spiral? We begin by writing its constant form as

$$\frac{a - \kappa^2}{\kappa'^2} = \pm c^2. \quad (19)$$

Now for  $a = 0$ , the solution will be Nielsen's spiral, while for other  $a$  values we get trig-aesthetic curves. With another derivation step we can eliminate the  $c$  parameter:

$$\kappa^2 - \frac{\kappa\kappa'^2}{\kappa''} = a. \quad (20)$$

This separates the two (although somewhat redundantly), but we can no longer select the shape inside the trig-aesthetic family.

Due to the above considerations, we can conclude that complex trig-aesthetic curves are indeed more closely connected to trig-aesthetic curves than to Nielsen's spiral, even though their shape is more similar to the latter. Reasons include:

- The common constant form  $\kappa/\kappa''$  is not necessarily a shape parameter. (It is for the general case, but for Nielsen's spiral it is only a scaling factor.)
- Nielsen's spiral is seamlessly integrated into the log-aesthetic family by the constant form  $\kappa\kappa''/\kappa'^2$ , with value 1.
- Complex trig-aesthetic curves are seamlessly integrated into the trig-aesthetic family by negative values of the constant form  $(1 - \kappa^2)/\kappa'^2$ .
- Both the combination and separation of these families are questionable.

### 2.1. The LAC-TAC Common Form

Let us take a closer look at the constant form in Eq. (18). We can rewrite it as

$$\frac{(\ln \kappa'')'}{(\ln \kappa)'} = \hat{C}, \quad (21)$$

or equivalently as

$$(\ln \kappa'')' = \hat{C}(\ln \kappa)', \quad (22)$$

which we integrate to obtain

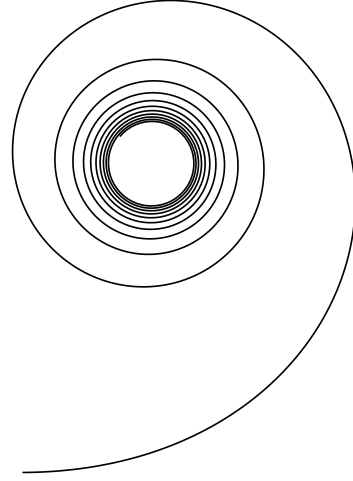
$$\ln \kappa'' = \hat{C} \ln \kappa + \ln \hat{A}, \quad (23)$$

where  $\ln \hat{A}$  is the integration constant ( $\hat{A} \neq 0$ ). Taking the exponential, we get

$$\kappa'' = \hat{A} \kappa^{\hat{C}}. \quad (24)$$

Multiplying both sides by  $2\kappa'$  and integrating once again results in

$$\kappa'^2 = \frac{2\hat{A}}{\hat{C}+1} \kappa^{\hat{C}+1} + \hat{B}, \quad (25)$$



**Figure 2:** The curve defined by  $\kappa' = \sqrt{\frac{1}{\kappa} + 1}$ , which does not have a closed-form representation.

where  $\hat{B}$  is the integration constant. A somewhat simplified form is

$$\kappa'^2 = A\kappa^B + C. \quad (A, B \neq 0) \quad (26)$$

Generally this involves elliptic or hypergeometric integrals, but there are closed-form solutions for special cases:

- For  $B = 1$ , the curvature is quadratic:

$$\kappa(s) = as^2 + bs + c, \quad (27)$$

where  $A = 4a$  and  $C = b^2 - 4ac$ . These are special cases of the generalized catenaries.<sup>5</sup>

- For  $C = 0$ , we have log-aesthetic curves of the form

$$\kappa(s) = (c_0s + c_1)^{-\frac{1}{\alpha}}, \quad (28)$$

with  $\alpha = (B-2)/2$ ,  $c_0 = -\alpha\sqrt{A}$ ,  $c_1$  arbitrary. In the  $B = 2$  case we get back Nielsen's spiral

$$\kappa(s) = c_0 \exp(\sqrt{A}s), \quad (29)$$

with arbitrary  $c_0$ .

- For  $B = 2$ ,  $A = -C$ , we get trig-aesthetic curves

$$\kappa(s) = \cos(s/c), \quad (30)$$

where  $c = 1/\sqrt{C}$ . Note that the complex-hyperbolic variant occurs when  $C < 0$ .

- Other cases, while not as simple, can easily be plotted (see e.g. Fig. 2), thanks to the ODE form

$$\kappa' = \pm \sqrt{A\kappa^B + C} \quad (31)$$

coming from Eq. (26). The sign should be chosen with care (flipping at the extrema for curves with  $AC < 0$ ).

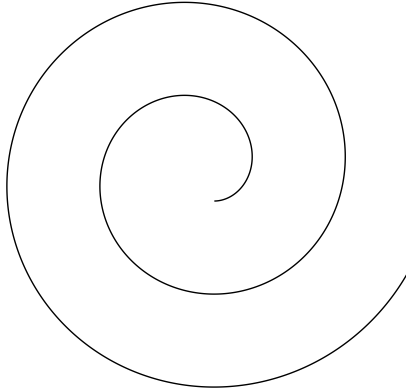


Figure 3: Output of the program in Fig. 4.

### Conclusion

We have shown the invariants for some representative aesthetic curve families, and also proposed the use of a simpler ‘constant form’. The application of these concepts was demonstrated through the classification of complex trig-aesthetic curves.

A common form of log- and trig-aesthetic curves was also presented. The investigation of other curves in this family is left for future research.

### Acknowledgements

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### References

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2. Raphael Linus Levien. The elastica – A mathematical history. Technical Report UCB/EECS-2008-103, University of California, Berkeley, 2008.
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4. Péter Salvi. The curves of Eduard Lehr. *G – Slovak Journal for Geometry and Graphics*, 22(43):5–23, 2025.
5. Péter Salvi. Generalized catenaries and trig-aesthetic curves. *Computer-Aided Design and Applications*, 23(1):56–67, 2026.

### Appendix A: Snippet for drawing curves

Figures 2 and 3 were generated by a simple POSTSCRIPT snippet shown in Figure 4. It can draw the curve families

```

%!PS

% Parameters
/origin_x 250 def
/origin_y 300 def
/s_max 60 def
/scale 30 def
/resolution 100 def

/scale scale resolution div def
/deg { 180 mul 3.1415926535 div } def

newpath origin_x origin_y moveto

1 0 % kappa(0) theta(0)
0 1 s_max resolution mul {
  pop 1 index resolution div add exch
  % -----
  % Signature: ( theta kappa -- theta kappa dkappa )
  % -----

  %%%
  % 1. Log-aesthetic curve
  %%%
% 2 % shape parameter alpha != 0
% dup 2 index exch 1 add exp exch div -1 mul

  %%%
  % 2. Nielsen's spiral (log-aesthetic with alpha = 0)
  %%%
% dup

  %%%
  % 3. Elastica
  %%%
% 0.5 % shape parameter lambda
% 2 index deg sin mul -1 mul

  %%%
  % 4. Trig-aesthetic curve
  %%%
% 1.5 % shape parameter c
% dup mul 2 index exch div -1 mul

  %%%
  % 5. Complex trig-aesthetic curve
  %%%
% 1 % shape parameter c
% dup mul 2 index exch div

  %%%
  % 6. LAC-TAC Common Form
  %%%
% 1 1 -1 % C A B
% 3 index exch exp mul add sqrt

  % -----
  resolution div add % theta kappa
  exch dup deg cos 1 index deg sin % kappa theta x y
  scale mul exch scale mul exch rlineto
} for
pop pop

stroke showpage

```

Figure 4: POSTSCRIPT snippet for drawing curves.

discussed in the paper, but it is easy to modify it to support others, as well. The source code is embedded in the PDF version of this document.